

1948

# The Field of Values of a Matrix.

John Cecil Currie

*Louisiana State University and Agricultural & Mechanical College*

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**THE FIELD OF VALUES OF A MATRIX**

**A Dissertation**

**Submitted to the Graduate Faculty of the  
Louisiana State University and  
Agricultural and Mechanical College  
in partial fulfillment of the  
requirements for the degree of  
Doctor of Philosophy**

**in**

**The Department of Mathematics**

**by**

**John Cecil Currie**

**B. S., Mississippi Southern College, 1933**

**M. A., University of Mississippi, 1936**

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## ABSTRACT

Let  $A = (a_{ij})$  be an  $m$ -rowed square matrix whose elements are complex numbers; let  $x = (x_1, x_2, \dots, x_m)$  be a unit vector, which means that  $xx' = \sum_{i=1}^m x_i \bar{x}_i = 1$ . Then the matrix product  $xAx' = \sum_{i,j} a_{ij} x_i \bar{x}_j$  is a one by one matrix, regarded as identical with the complex number which constitutes its single element. The set of numbers  $xAx'$  for a given matrix  $A$ , with  $x$  ranging over the set of unit vectors, is called the field of values of the matrix  $A$ , and is regarded, geometrically, as forming a point set in the complex plane. The concept was introduced in 1918 by O. Toeplitz, who proved that the field of values lies within a rectangle with sides parallel to the real and imaginary axes. A little later, Hausdorff showed that the field of values is closed, bounded, connected, and convex. The earlier sections of this dissertation give a summary of these results of Toeplitz and Hausdorff. In view of the greater convenience of studying the field of values of a matrix in triangular form, there is also a discussion of this form.

With the exception of a paper by Murnaghan in 1932, the published work on the field of values of a matrix has used the concept chiefly as a source of theorems on limits for the characteristic roots of the matrix. The present

study lays aside this emphasis on characteristic roots in order to examine the nature of the boundary of the field of values. This boundary is shown to be a portion (or, for a second order matrix, all) of a certain plane algebraic curve. A general method is described whereby the Cartesian equation of the curve may be calculated, and the procedure is carried out in some detail for the second, third, and fourth order cases. An analysis with the help of the Plücker formulas leads to a classification of the different types of curves which arise from different types of matrices. For a matrix of order  $m$ , the associated curve is of class  $m$ , degree  $m(m - 1)$ ; it has no inflection points, and has  $m$  real foci, which are in fact, the characteristic roots of the matrix.

A different type of equation is devised in the paper by Murnaghan already referred to. The variables in the equation are the distances from the foci to the tangents to the curve. Murnaghan calculated in detail the results in the second and third order cases; in the present work, the corresponding result in the fourth order is derived and discussed.

In the concluding sections, the effect of a collineatory transformation on the field of values is examined; some observations are made on certain special types of matrices, and on some related sets of points.



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# I

## Definitions and Notation

1. Rectangular matrices. A matrix is a set of  $mn$  elements arranged in  $m$  rows and  $n$  columns, thus:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

in which the symbol  $a_{ij}$  denotes the element in the  $i$ -th row and  $j$ -th column. The customary abbreviation of the array above is  $A = (a_{ij})$ . The elements  $a_{ij}$  constitute the principal diagonal of the matrix.

The elements of the matrices to be discussed are complex numbers. If  $a$  is a complex number, the conjugate complex number is denoted by  $\bar{a}$ . Then if  $A = (a_{ij})$ , the matrix  $(\bar{a}_{ij})$  is denoted by  $\bar{A}$ .

2. Algebraic operations on matrices. If  $A = (a_{ij})$  and  $B = (b_{ij})$  are matrices with  $m$  rows and  $n$  columns,  $\lambda$  and  $\mu$  are complex numbers, then the quantity  $\lambda A + \mu B = (c_{ij})$ , where  $c_{ij} = \lambda a_{ij} + \mu b_{ij}$ , for  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, n$ .

If  $A = (a_{ij})$  is an  $m$  by  $n$  matrix and  $B = (b_{ij})$  is

an  $n$  by  $s$  matrix, then the product  $AB = (c_{ij})$ , where

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} \text{ for } i = 1, 2, \dots, m, \text{ and } j = 1, 2, \dots, s.$$

Thus, the product is an  $m$  by  $s$  matrix, and multiplication is defined if, and only if, the number of columns of the matrix on the left is equal to the number of rows of the matrix on the right.

3. The transpose of a matrix. The transpose of a matrix, denoted by  $A'$ , is the matrix obtained by interchanging the rows and columns of  $A$ . Thus, if  $A = (a_{ij})$ , and  $A' = (a'_{ij})$ , then  $a'_{ij} = a_{ji}$  for  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, m$ .

4. Special types of matrices. A square matrix is a matrix in which  $m = n$ . With the exception of vectors, defined below, the discussion henceforth is confined to square matrices. A triangular matrix is a matrix in which all the elements on one side of the principal diagonal are zero. Triangular matrices are of two types, according as the zero elements are below (type 1) or above (type 2) the principal diagonal. For simplicity, the term triangular matrix will be assumed to refer to type 1 unless otherwise specified.

If the only non-zero elements of a matrix are in the principal diagonal, the matrix is called a diagonal matrix and is denoted by  $\text{diag} \{a_1, a_2, \dots, a_m\}$ , where  $a_i$  denotes the element in the  $i$ -th row and  $i$ -th column.

If, in a diagonal matrix, each diagonal element is equal to  $a$ , the matrix is called a scalar matrix; and if  $a = 1$ , it is called the identity matrix, denoted by  $I$ .

If each element of a matrix is zero, it is called the zero matrix, and is denoted by  $O$ .

If for a given matrix  $A$  there exists a matrix  $B$  such that  $AB = I$ , then  $B$  is called the inverse of  $A$ , and is denoted by  $A^{-1}$ . It may be shown that  $A^{-1}$  is unique, and that  $A^{-1}A = I$ . A matrix has an inverse if and only if the determinant of the matrix is different from zero. The determinant of a matrix (defined only for a square matrix) is formed by regarding the array constituting the matrix as a determinant and is denoted by  $|A|$ . The usual properties and definitions in determinant theory are assumed.

A matrix  $A$  with the property that  $A\bar{A}' = I$  is called unitary. Since also  $\bar{A}'A = I$ , the matrix  $\bar{A}'$  is unitary.

If  $A = A'$ , the matrix is called a symmetric matrix. If all the elements of a symmetric matrix are real, it is said to be real symmetric. If  $A = \bar{A}'$ , the matrix is called Hermitian. If all the elements of an Hermitian matrix are real, it is evidently a real symmetric matrix.

If  $A$  is commutative with  $\bar{A}'$  ( $A\bar{A}' = \bar{A}'A$ ) then  $A$  is called a normal matrix. It follows from the preceding definition that an Hermitian matrix is normal.

A row vector is a matrix with one row; a column

vector is a matrix with one column. For simplicity, a row vector will usually be referred to as a vector, and a column vector regarded as the transpose of a vector. Vectors are denoted by small letters, as  $x = (x_1, x_2, \dots, x_m)$ . The prime and conjugate symbols are reserved to column vectors so that the vector product  $x\bar{y}'$  is a one by one matrix, whereas  $\bar{y}'x$  is an  $m$  by  $m$  matrix. No distinction is made between the one by one matrix ( $a$ ) and the complex number  $a$ . If  $x\bar{x}' = a$ , then  $\sqrt{a}$  is called the norm of  $x$ , and if  $a = 1$ , the vector is called a unit vector. Any non-zero vector becomes a unit vector if each of its elements is divided by the norm of the vector.

A column vector  $\bar{x}'$  with the property that  $A\bar{x}' = 0$  is called a right annihilator of the matrix  $A$ .

5. The characteristic function of a matrix. The characteristic function of a matrix  $A$  is the determinant  $|\lambda I - A|$ , where  $\lambda$  is an indeterminate. The characteristic equation of  $A$  is  $|\lambda I - A| = 0$ . The characteristic roots of  $A$  are the roots of the characteristic equation.

## II

### The Field of Values of a Matrix

1. Definition of the field of values. Let  $x$  be a unit vector and  $A$  be a square matrix. Then the matrix product  $x\bar{A}x'$  is a one by one matrix; i. e., a complex number. For a given matrix  $A$ , let  $x$  range over the set of all unit vectors. Then the complex numbers  $x\bar{A}x'$  form a set of points in the complex plane. This set of points is called the field of values of  $A$  and is denoted by  $W(A)$ .

This concept was first introduced by von Toeplitz,<sup>1</sup> who used the term Wertvorrat. He established that  $W(A)$  is contained within a bounded convex region; a little later, Hausdorff proved that  $W(A)$  is itself a bounded, closed, connected, convex region.<sup>2</sup> A still later paper by Murnaghan gave additional information on the nature of  $W(A)$ , particularly in the second and third order cases.<sup>3</sup> Interest in

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<sup>1</sup> O. Toeplitz, "Das algebraische Analogon zu einem Satze von Fejer," Mathematische Zeitschrift, II (1918), 187-197.

<sup>2</sup> Felix Hausdorff, "Der Wertvorrat einer Bilinearform," Mathematische Zeitschrift, III (1919), 314-316.

<sup>3</sup> F. D. Murnaghan, "On the Field of Values of a Square Matrix," Proceedings of the National Academy of Sciences, XVIII (1932), 246-248.

the concept has been due principally to the fact that it has been used in connection with the problem of determining limits for the characteristic roots of a matrix. Several writers (See, for example, a paper by A. B. Farnell)<sup>4</sup> have given limits for the characteristic roots which are actually limits for the field of values. The fact that such limits are limits for the characteristic roots is the burden of the following theorem.

Theorem 2.1. The characteristic roots of a matrix belong to its field of values.

A system of  $m$  linear homogeneous equations in  $m$  unknowns has a non-trivial solution if and only if the determinant of the system is zero. Or, in matrix terminology, a necessary and sufficient condition for the existence of a right annihilator of a matrix is that the determinant of the matrix is zero. Thus, there exists a column vector  $\bar{x}'$  such that  $(\lambda I - A)\bar{x}' = 0$  if and only if  $\lambda$  is a characteristic root of  $A$ . The vector may be assumed to be a unit vector, since division of the above equation by the norm of  $x$  would not change the equality. Then  $x(\lambda I - A)\bar{x}' = 0$

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<sup>4</sup> A. B. Farnell, "Limits for the Characteristic Roots of a Matrix," Bulletin of the American Mathematical Society, L (1944), 789-794.



results from multiplying on the left by  $x$ . Now  $x\lambda Ix' = \lambda$ ; therefore,  $x\bar{A}x' = \lambda$ , and  $\lambda$  belongs to  $W(A)$ .

2. Rotations and translations of  $W(A)$ . If  $t = e^{i\theta}$ , then the field of values of  $\bar{t}A$  is obtained by rotating the field of values of  $A$  about the origin through an angle  $-\theta$ . For, if  $x\bar{A}x'$  is any point of  $W(A)$ , then  $x(\bar{t}A)\bar{x}' = \bar{t}(x\bar{A}x')$  is the corresponding point of  $W(\bar{t}A)$ . However, it is convenient to regard the rotation of  $W(A)$  through an angle  $-\theta$  as leaving the field of values unchanged in position but as referring it to  $\xi$  and  $\eta$  axes which make an angle  $\theta$  with the  $x$  and  $y$  axes of the complex plane so that a point  $x + iy$  in the  $XY$  plane is a point  $\xi + i\eta$  in the  $\xi\eta$  plane, where

$$x = \xi \cos \theta - \eta \sin \theta$$

$$y = \xi \sin \theta - \eta \cos \theta$$

If  $c$  is a complex number, then the field of values of  $A - cI$  is obtained by translating the field of values of  $A$  through a distance equal to the absolute value of  $c$  and in the direction of  $-c$ . For, if  $x\bar{A}x'$  is a point of  $W(A)$ , then  $x(A - cI)\bar{x}' = x\bar{A}x' - c$  is the corresponding point of  $W(A - cI)$ .

3. Standard position of  $W(A)$ . If the characteristic equation of a matrix  $A$  is

$$\lambda^m - p_1 \lambda^{m-1} + p_2 \lambda^{m-2} + \dots + (-1)^i p_i \lambda^{m-i} + \dots + (-1)^m p_m = 0$$

then  $p_i$  is the sum of the  $i$ -th order principal minors of

A.<sup>5</sup> (Principal minors: minors whose principal diagonal elements are principal diagonal elements of A). Then  $p_1 = \sum_{i=1}^m a_{i,i}$ , and since also  $p_1 = \sum_{i=1}^m \lambda_i$ , we have the theorem:

**Theorem 2.2.** The centroid of the characteristic roots of a matrix is identical with the centroid of the elements of the principal diagonal.

The field of values,  $W(A)$ , of a matrix will be said to be in standard position if  $\sum_{i=1}^m a_{i,i} = 0$ . If  $W(A)$  is not in standard position, form the matrix  $B = A - (\sum_{i=1}^m a_{i,i}/m)I$ , so that  $W(B)$  is in standard position. This translation alters only the position of the field of values, the discussion of which is simpler in some respects if it is in standard position; therefore, in the present study, a matrix is ordinarily assumed to have its field of values in standard position.

This preliminary section on fields of value is concluded here in order to insert a section on some of the properties of triangular matrices, unit vectors, and unitary matrices.

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<sup>5</sup> C. C. Macduffee, The Theory of Matrices (corrected reprint of first edition; New York: Chelsea Publishing Company, 1946), p. 19.

### III

#### The Triangular Matrix

1. On unitary matrices. Let  $u_i = (u_{i1}, u_{i2}, \dots, u_{im})$  be the  $i$ -th row of a unitary matrix  $U$ . Then, since  $U\bar{U}' = I$ , it follows that  $u_i\bar{u}_i' = 1$ . This shows that each row of a unitary matrix is a unit vector. Also,  $u_i\bar{u}_j' = 0$ ; two unit vectors so related are said to be conjugate-orthogonal. Obviously the same properties hold for columns of a unitary matrix.

It may be noted that the product of two unitary matrices is unitary, for, if  $U$  and  $V$  are unitary,  $UV\bar{V}'\bar{U}' = U\bar{U}' = I$ .

2. On unit vectors. For any unit vector  $x$  and any unitary matrix  $U$ , the vector  $y = xU$  is a unit vector, since  $y\bar{y}' = xU\bar{U}'\bar{x}' = x\bar{x}' = 1$ . Any two unit vectors are connected by a relationship of this kind, according to a theorem of W. V. Parker's:

Theorem 3.1. For any two unit vectors  $x$  and  $y$  there exists a unitary matrix  $U$  such that  $y = xU$ .

Let  $e_1$  be the unit vector  $(1, 0, \dots, 0)$ . Then  $x = e_1V$  where  $V$  is a unitary matrix whose first row is  $x$ . Similarly,  $y = e_1W$ , where  $W$  is a unitary matrix whose first row is  $y$ . Then  $e_1 = x\bar{V}'$ , and  $y = x\bar{V}'W$ .

That a unitary matrix may be constructed with an arbitrary unit vector as the first row may be shown as follows: Let  $u_1 = (u_{11}, u_{12}, \dots, u_{1m})$  be the given vector; then  $\bar{u}_1'$  is the first column of  $\bar{U}'$ . The 1-th column of  $\bar{U}'$  is sought as the solution of the system of equations  $u_1 \bar{u}_1' = 0, u_2 \bar{u}_1' = 0, \dots, u_{i-1} \bar{u}_1' = 0$ . A non-trivial solution exists, since there are more unknowns than equations. The resulting set of solutions for  $i = 2, 3, \dots, m$  completes the construction of  $\bar{U}'$ , from which  $U$  may be written.

3. The triangular matrix. Since a matrix in triangular form exhibits its characteristic roots in the principal diagonal, this form is the most convenient for studying the field of values. A matrix not in triangular form may be unitarily transformed into a triangular matrix, according to the following theorem:

Theorem 3.2. For any matrix  $A$  there exists a unitary matrix  $U$  such that  $UA\bar{U}'$  is triangular.<sup>1</sup>

Let  $\bar{u}_1'$  be a unit vector such that  $A\bar{u}_1' = \lambda_1 \bar{u}_1'$ ,  $\lambda_1$  a characteristic root of  $A$ . Construct a unitary matrix  $U_1$  with  $u_1$  as its first row. Let  $B_1 = U_1 A \bar{U}_1'$ . Then  $B_1$  is

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<sup>1</sup> I. Schur, "Über die charakteristischen Wurzeln einer linearen Substitution," Mathematische Annalen, LXVI (1909), 488-510.

of the form

$$B_1 = \begin{pmatrix} \lambda_1 & b_{12} & b_{13} & \dots & b_{1m} \\ 0 & b_{22} & b_{23} & \dots & b_{2m} \\ 0 & b_{32} & b_{33} & \dots & b_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & b_{m2} & b_{m3} & \dots & b_{mm} \end{pmatrix}$$

For, let  $u_i$  be the  $i$ -th row of  $U_1$ ; then the first column of  $B_1$  consists of elements  $u_i A \bar{u}_1' = u_i \lambda_1 \bar{u}_1' = \lambda_1 u_i \bar{u}_1'$ . For  $i = 1$ , this is  $\lambda_1$ ; for  $i > 1$ , this is zero, since the rows of  $U$  are conjugate-orthogonal. The  $(m - 1)$ th order submatrix obtained by omitting the first row and first column of  $B_1$  has as its characteristic roots the remaining characteristic roots  $\lambda_2, \lambda_3, \dots, \lambda_m$  of  $A$ , since  $A$  and  $B_1$  are similar.<sup>2</sup> Thus the same process applied to this submatrix,  $C$ , of  $B_1$  yields an  $(m - 1)$ th order unitary matrix  $V$  such that  $VC\bar{V}'$  is of the form

$$\begin{pmatrix} \lambda_2 & c_{12} & \dots & c_{1m-1} \\ 0 & c_{22} & \dots & c_{2m-1} \\ \dots & \dots & \dots & \dots \\ 0 & c_{m-1,2} & \dots & c_{m-1,m-1} \end{pmatrix}$$

Then the matrix

$$U_2 = \begin{pmatrix} 1 & 0 \\ 0 & V \end{pmatrix}$$

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<sup>2</sup> Macduffee, op. cit., p. 69.

is unitary of the  $m$ -th order, and  $B_2 = U_2 B_1 \bar{U}_2' = U_2 U_1 A \bar{U}_1' \bar{U}_2'$  is of the form

$$B_2 = \begin{pmatrix} \lambda_1 & d_{12} & d_{13} & \dots & d_{1m} \\ 0 & \lambda_2 & d_{23} & \dots & d_{2m} \\ 0 & 0 & d_{33} & \dots & d_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & d_{m3} & \dots & d_{mm} \end{pmatrix}$$

An induction may be established to obtain a sequence  $U_1, U_2, \dots, U_{m-1}$  of unitary matrices such that if  $U = U_{m-1} U_{m-2} \dots U_1$ , then  $UA\bar{U}'$  is triangular.

The effect of this transformation on a normal matrix is observed as a corollary. It should be noted that a unitary transform of a normal matrix is normal, for if  $B = UA\bar{U}'$ , then  $B\bar{B}' = UA\bar{U}'U\bar{A}'\bar{U}' = UA\bar{A}'\bar{U}'$ , and  $\bar{B}'B = U\bar{A}'\bar{U}'UA\bar{U}' = U\bar{A}'A\bar{U}'$ , so that, if  $A\bar{A}' = \bar{A}'A$ , then  $B\bar{B}' = \bar{B}'B$ .

**Corollary 3.2.** A necessary and sufficient condition that a matrix be normal is that it be a unitary transform of a diagonal matrix.<sup>3</sup>

Let  $A$  be normal; then by the preceding it is a unitary transform of a normal, triangular matrix  $B$ .

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<sup>3</sup> Toeplitz, op. cit., 187-197.

$$B = \begin{pmatrix} \lambda_1 & b_{12} & \dots & b_{1m} \\ 0 & \lambda_2 & \dots & b_{2m} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_m \end{pmatrix}$$

The element in the first row and first column of  $B\bar{B}'$  is  $\lambda_1 \bar{\lambda}_1 + b_{12} \bar{b}_{12} + b_{13} \bar{b}_{13} + \dots + b_{1m} \bar{b}_{1m}$ , and the corresponding element of  $\bar{B}'B$  is  $\lambda_1 \bar{\lambda}_1$ . Hence,  $b_{12} = b_{13} = \dots = b_{1m} = 0$ . Similarly, elements in the second row and second column may now be compared to show that  $b_{23} = b_{24} = \dots = b_{2m} = 0$ , and an induction established to complete the proof that  $B$  is diagonal. The sufficiency is evident, since diagonal matrices are commutative.

4. On uniqueness of the triangular form. Consideration of the possibilities of the triangular matrix as a canonical form under unitary transformations led to the following result in the case in which the characteristic roots are distinct.

Theorem 3.3. If  $A$  is a matrix with distinct characteristic roots, and  $B$  and  $C$  are triangular matrices, each of which is a unitary transform of  $A$ , with characteristic roots in the same order, then (1) the unitary matrix which transforms  $B$  into  $C$  is diagonal; (2) for each  $i$  and  $j$ ,  $|b_{ij}| = |c_{ij}|$ .

Let  $B = UA\bar{U}'$  be triangular, as given above in the proof of Corollary 3.2, and assume the  $\lambda_i$  are distinct.

For each  $k = 1, 2, \dots, m$  there exists a unit vector  $v_k$  such that  $B\bar{v}_k' = \lambda_k \bar{v}_k'$ ; and since  $B - \lambda_k I$  is of rank  $m - 1$ ,  $v_k$  is unique except for a factor of unit modulus.<sup>4</sup> Now suppose  $C = VB\bar{V}'$  is also triangular:

$$C = \begin{pmatrix} \lambda_1 & c_{12} & \dots & c_{1m} \\ 0 & \lambda_2 & \dots & c_{2m} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_m \end{pmatrix}$$

Denote the rows of  $V$  by  $v_1, v_2, \dots, v_m$ . Then  $VB\bar{v}_1'$  is the first column of  $C$ , from which  $B\bar{v}_1' = \lambda_1 \bar{v}_1'$ . Hence  $v_1 = (e^{i\theta_1}, 0, \dots, 0)$ , and

$$V = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & V_1 \end{pmatrix}$$

Write

$$B = \begin{pmatrix} \lambda_1 & b \\ 0 & B_1 \end{pmatrix}, \quad C = \begin{pmatrix} \lambda_1 & c \\ 0 & C_1 \end{pmatrix}$$

so that

$$C = VB\bar{V}' = \begin{pmatrix} \lambda_1 & e^{i\theta_1} b \bar{v}_1' \\ 0 & V_1 B_1 \bar{V}_1' \end{pmatrix}$$

Hence,  $V_1 B_1 \bar{V}_1' = C_1$ , and by an exactly similar argument the first vector in  $V_1$  is  $(e^{i\theta_2}, 0, \dots, 0)$ , so that  $v_2 = (0, e^{i\theta_2}, 0, \dots, 0)$ . By induction,  $V$  is of the

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<sup>4</sup> L. E. Dickson, Modern Algebraic Theories (New York: Benj. H. Sanborn & Co., 1930), p. 62.



form diag  $\{ e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_m} \}$ . Therefore, for a particular order of the  $\lambda_i$ , the  $b_{ij}$  are unique except for a factor of unit modulus.

The proof follows an argument suggested by W. V. Parker.

If the characteristic roots are not all distinct, the theorem is no longer true as stated; for example, the matrices A and B shown below are unitary transforms of each other:

$$A = \begin{pmatrix} 1 & 0 & -9 \\ 0 & 1 & -3\sqrt{5} \\ 0 & 0 & -2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 3\sqrt{14} \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

It may be possible to devise some sort of unique triangular form for matrices with multiple roots by means of special restrictions on the  $b_{ij}$ .

## IV

### The Field of Values of a Normal Matrix

#### 1. The field of values of a unitary transform.

The special significance of the preceding section on unitary matrices and the triangular form becomes manifest in the following theorem:

Theorem 4.1. The field of values of a matrix  $A$  is identical with the field of values of any unitary transform of  $A$ .

Let  $y(UA\bar{U}')\bar{y}'$  be any point of  $W(UA\bar{U}')$ . Since matrix multiplication is associative, this product may be written  $(yU)A(\bar{U}'\bar{y}')$ . But because of Theorem 3.1 the range of the vector  $yU$  is identical with the range of an arbitrary vector  $x$ .

Corollary 4.1. For unitary matrices  $U$  and  $V$ ,  
 $W(UVA) = W(VAU) = W(AUV)$ .

The three matrices are unitary transforms of each other.

2. The field of values of a normal matrix. Corollary 3.2 in conjunction with Theorem 4.1 yields the fundamental result for normal matrices.

Theorem 4.2. The field of values of a normal matrix is the smallest convex polygon containing the

characteristic roots within or on the boundary.<sup>1</sup>

If  $A$  is normal, then a unitary transform of it is diagonal,  $\text{diag} \{ \lambda_1, \lambda_2, \dots, \lambda_m \}$ . Then  $W(A)$  is the set of points  $\lambda_1 x_1 \bar{x}_1 + \lambda_2 x_2 \bar{x}_2 + \dots + \lambda_m x_m \bar{x}_m$ . But this is simply the convex cover of the finite set of points  $\lambda_1, \lambda_2, \dots, \lambda_m$ .<sup>2</sup>

Corollary 4.2.1. The field of values of an Hermitian matrix is the segment of the real axis between its least and greatest characteristic roots.

An Hermitian matrix is normal and has only real roots;<sup>3</sup> the field of values is, therefore, the convex cover of a finite set of points on the real axis.

Corollary 4.2.2. The field of values of a unitary matrix is a polygon inscribed in the unit circle.

A unitary matrix is normal,<sup>4</sup> and its characteristic roots have unit moduli;<sup>5</sup> the field of values is, therefore, the convex cover of a finite set of points on the circumference of the unit circle.

<sup>1</sup> Toeplitz, op. cit., 187-197.

<sup>2</sup> Paul Alexandroff and Heinz Hopf, Topologie (Berlin: Julius Springer, 1935), I, p. 603.

<sup>3</sup> Macduffie, op. cit., p. 26.

<sup>4</sup> Ibid., p. 76.

<sup>5</sup> Ibid., p. 28.

## V

### The Field of Values of a Non-Normal Matrix

1. Hermitian decomposition of a matrix. A matrix may be written  $A = F + iG$ , where  $F$  and  $G$  are Hermitian; in fact,  $F = (A + \bar{A}')/2$  and  $G = (A - \bar{A}')/2i$ . Then  $F + iG$  is called the Hermitian decomposition of  $A$ .

For the Hermitian decomposition of  $\bar{t}A$  the following notation is used:  $\bar{t}A = F_\theta + iG_\theta$ . Let the characteristic roots of  $F_\theta$  be  $\xi_{\theta 1}, \xi_{\theta 2}, \dots, \xi_{\theta m}$ , and the characteristic roots of  $G_\theta$  be  $\eta_{\theta 1}, \eta_{\theta 2}, \dots, \eta_{\theta m}$ ; assume that the subscripts are so ordered that each of these sequences is monotonic decreasing.

2. The Toeplitz-Hausdorff theorem. Reference was made in II to the following fundamental theorem on the field of values of a matrix.

Theorem 5.1. The field of values of a matrix is a convex set. (A convex set is a set of points such that, if  $\alpha$  and  $\beta$  are two points of the set, then each point of the straight line segment joining  $\alpha$  and  $\beta$  is also in the set).

Let  $\bar{t}A = F_\theta + iG_\theta$ . Since  $F_\theta$  and  $G_\theta$  are Hermitian, the numbers  $x F_\theta \bar{x}'$  and  $x G_\theta \bar{x}'$  are real; then  $x(\bar{t}A)\bar{x}'$  is a complex number whose real part is  $x F_\theta \bar{x}'$  and whose imaginary part is  $x G_\theta \bar{x}'$ . Therefore,  $W(A)$  is a subset of the rectangle

whose boundaries in the  $\xi\eta$  plane are the lines

$\xi = \xi_{01}$ ,  $\xi = \xi_{0m}$ ,  $\eta = \eta_{01}$ ,  $\eta = \eta_{0m}$  and this rectangle is evidently the smallest rectangle with sides parallel to the  $\xi\eta$  axes for which this statement can be made.

Assume that  $F_\theta$  is diagonal, which assumption is possible because of Corollary 3.2. Let  $u_0$  be a point of  $W(F_\theta)$ , and let  $M$  be the set of vectors  $x$  such that  $x F_\theta \bar{x}' = u_0$ . If  $A$  is an  $m$ -th order matrix, these vectors may be regarded as points in real  $2m$  space, and that this set of points in  $2m$  space is closed and bounded follows from the two conditions imposed on  $x$  ( $x\bar{x}' = 1$ , and  $x F_\theta \bar{x}' = u_0$ ).<sup>1</sup> Hausdorff's argument that  $M$  is connected is as follows:

Let  $x$  and  $y$  be two vectors of  $M$ ; for each  $j = 1, 2, \dots, m$ , let  $|x_j| = p_j$ ,  $x_j = p_j e^{i\alpha_j}$ , and  $|y_j| = q_j$ ,  $y_j = q_j e^{i\beta_j}$ . Let  $p = (p_1, p_2, \dots, p_m)$ ,  $q = (q_1, q_2, \dots, q_m)$ . The  $p$  and  $q$  also belong to  $M$ . The four points  $x, p, q, y$  in  $2m$  space are connected in  $M$  as the parameter  $s$  varies from 0 to 1 by variable  $z$  defined as follows: to join  $x$  to  $p$ , let  $z_j = p_j e^{(1-s)i\alpha_j}$ ; to join  $p$  to  $q$ , let  $z_j = \sqrt{(1-s)p_j^2 + sq_j^2}$ ; to join  $q$  to  $y$ , let  $z_j = q_j e^{is\beta_j}$ ; in each case,  $j = 1, 2, \dots, m$ .

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<sup>1</sup> M. H. A. Newman, Elements of the Topology of Plane Sets of Points (Cambridge: Cambridge University Press, 1939), p. 26.

The vectors  $z = (z_1, z_2, \dots, z_m)$  defined in this fashion are in  $M$ , since  $zF_\theta \bar{z}' = u_\theta$ . Thus,  $M$  is a closed, bounded, connected set, which implies that the set of real numbers  $xG_\theta \bar{x}'$ ,  $x$  in  $M$ , is closed, bounded, and connected; that is, either a straight line segment or a single point. Therefore, the points common to  $W(A)$  and the line  $\xi = u_\theta$  consist of a straight line segment or a single point. Since  $\theta$  is arbitrary, it follows that  $W(A)$  is convex.

If  $\xi_{\theta 1}$  is a simple root, then the set  $M$  of vectors  $x$  for which  $xF_\theta \bar{x}' = \xi_{\theta 1}$  consists of the vectors  $(e^{i\alpha}, 0, \dots, 0)$  for  $0 \leq \alpha \leq 2\pi$ ; but for these vectors,  $xG_\theta \bar{x}'$  is unique, so that the bounding line  $\xi = \xi_{\theta 1}$  contains exactly one point of  $W(A)$ .

If  $\xi_{\theta 1}$  is a multiple root, say of order  $r$ , so that  $\xi_{\theta 1} = \xi_{\theta 2} = \dots = \xi_{\theta r} \geq \xi_{\theta r+1} \geq \xi_{\theta r+2} \geq \dots \geq \xi_{\theta m}$ , then the set  $M$  of vectors  $x$  for which  $xF_\theta \bar{x}' = \xi_{\theta 1}$  consists of vectors  $x = (x_1, x_2, \dots, x_r, 0, 0, \dots, 0)$ , and hence  $xG_\theta \bar{x}'$ , for  $x$  in  $M$ , is simply an Hermitian form of order  $r$ , assuming all values on a straight line segment. Thus, if  $\xi_{\theta 1}$  is a multiple root, the bounding line  $\xi = \xi_{\theta 1}$  has a straight line segment in common with  $W(A)$ .

3. The boundary of the field of values. Let  $A$  be the triangular matrix

$$A = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} & \dots & a_{1m} \\ 0 & \lambda_2 & a_{23} & \dots & a_{2m} \\ 0 & 0 & \lambda_3 & \dots & a_{3m} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_m \end{pmatrix}$$

Assume that  $\sum_{i=1}^m \lambda_i = 0$ . Let  $F_\theta + iG_\theta$  be the Hermitian decomposition of  $\bar{t}A$ ; then

$$F_\theta = \begin{pmatrix} \xi - (\lambda_1 \bar{t} + \bar{\lambda}_1 t)/2 & (a_{12} \bar{t})/2 & \dots & (a_{1m} \bar{t})/2 \\ (\bar{a}_{12} t)/2 & \xi - (\lambda_2 \bar{t} + \bar{\lambda}_2 t)/2 & \dots & (a_{2m} \bar{t})/2 \\ \dots & \dots & \dots & \dots \\ (\bar{a}_{1m} t)/2 & (\bar{a}_{2m} t)/2 & \dots & \xi - (\lambda_m \bar{t} + \bar{\lambda}_m t)/2 \end{pmatrix}$$

The characteristic equation of  $F_\theta$  is

$$\begin{vmatrix} \xi - (\lambda_1 \bar{t} + \bar{\lambda}_1 t)/2 & -(a_{12} \bar{t})/2 & \dots & -(a_{1m} \bar{t})/2 \\ -(a_{12} t)/2 & \xi - (\lambda_2 \bar{t} + \bar{\lambda}_2 t)/2 & \dots & -(a_{2m} \bar{t})/2 \\ \dots & \dots & \dots & \dots \\ -(a_{1m} t)/2 & -(a_{2m} t)/2 & \dots & \xi - (\lambda_m \bar{t} + \bar{\lambda}_m t)/2 \end{vmatrix} = 0$$

Since  $F_\theta$  is Hermitian, the roots of this equation are all real; the  $m$  real lines  $\xi = \xi'_{\theta j}$ , for  $j = 1, 2, \dots, m$ , form a family with  $\theta$  as parameter, the envelope of which is necessarily a class  $m$  curve,<sup>2</sup> which will be denoted by  $C(A)$ . Thus, for each  $\theta$ ,  $0 \leq \theta \leq 2\pi$ , there are  $m$  real

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<sup>2</sup> H. Hilton, Plane Algebraic Curves (Second edition; London: Oxford University Press, 1932), p. 58.

lines, each tangent to  $C(A)$  at one point. There may be a value (or values) of  $\theta$  for which  $F_\theta$  has a multiple root, say of order  $r$ ; then, for this  $\theta$ ,  $r$  of the  $m$  lines coincide, so that the resulting line is tangent to  $C(A)$  at  $r$  points, which may or may not be distinct. It is usually more convenient to regard such a line as  $r(r-1)/2$  coincident bitangents (a bitangent is a line tangent to a curve at two points). Though such lines are referred to as bitangents to  $C(A)$ , they are also a part of  $C(A)$  themselves; as the argument for this is a little complicated, I state it as a theorem.

**Theorem 5.2.** The bitangents to  $C(A)$  belong to  $C(A)$ .

From the above discussion it is evident that a necessary and sufficient condition for  $C(A)$  to have a bitangent is for  $F_\theta$  to have a double root.

A more precise description is required of the manner in which the  $m$  real lines described above "envelope"  $C(A)$ . Let  $\xi_{\theta j}$  be a simple root of  $F$ , and let  $\bar{x}'$  be the characteristic unit vector of  $\xi_{\theta j}$ ; that is,  $\bar{x}'$  is the solution of  $F_\theta \bar{x}' = \xi_{\theta j} \bar{x}'$ . This vector is unique; if  $F_\theta$  is assumed to be diagonal, it is, in fact,  $(0, 0, \dots, e^{i\alpha}, 0, \dots, 0)$ , where the  $e^{i\alpha}$  is in the  $j$ -th column. Then  $x(\bar{t}A)\bar{x}'$  is the point of tangency of the line  $\xi = \xi_{\theta j}$  with  $C(A)$ , unique, since  $xG_\theta \bar{x}'$  is unique. On the other hand,



suppose  $\xi_{\theta j} = \xi_{\theta j+1}$  is a double root of  $F_{\theta}$ ; then characteristic vectors of  $\xi_{\theta j}$  are of the form  $x = (0, 0, \dots, x_j, x_{j+1}, 0, \dots, 0)$ , where  $x_j \bar{x}_j + x_{j+1} \bar{x}_{j+1} = 1$ , since  $x \bar{x}' = 1$ . Thus, for these vectors,  $x \bar{x}'$  is simply a Hermitian form of order 2, so that the numbers  $x \bar{x}'$  fill out a straight line segment. As a consequence, the numbers  $x(\bar{t}A)\bar{x}'$  fill out a straight line segment; that is, the line  $\xi = \xi_{\theta j}$  has a straight line segment in common with  $C(A)$ . The segment is, of course, the segment between the point of tangency of the line  $\xi = \xi_{\theta j}$  with  $C(A)$ , and the point of tangency (in general, distinct from the other) of the coincident line  $\xi = \xi_{\theta j+1}$  with  $C(A)$ . What has been shown is that each point of the segment between the two points of tangency of a bitangent of  $C(A)$  is "enveloped" in the same sense as any point of  $C(A)$ , and hence must also belong to  $C(A)$ . Furthermore, each such point is actually a double point, being on each of two lines, so that, if the Cartesian equation of the bitangent is  $ax + by + c = 0$ , then the Cartesian equation of  $C(A)$  contains the factor  $(ax + by + c)^2$ . More generally, if  $\xi_{\theta j}$  is a root of  $F_{\theta}$  of multiplicity  $r$ , then the line  $\xi = \xi_{\theta j}$  is counted as  $r(r-1)/2 = {}_r C_2$  bitangents; if the Cartesian equation of the line is  $ax + by + c = 0$ , then the Cartesian equation of  $C(A)$  contains  $(ax + by + c)^k$  as a factor, where  $k = 2 {}_r C_2$ .

Algebraically, when we say that  $ax + by + c$  is a factor of the Cartesian equation of a curve, this implies that all points of the line satisfy the equation. However, it is desired to restrict the notation  $C(A)$  to those points actually obtained as described above,  $x(\bar{t}A)\bar{x}' = \xi_{0j} + 1\eta$ , where  $\bar{x}'$  is a characteristic vector of  $\xi_{0j}$ . This restriction affects only the bitangents, which are thereby limited to those portions between their points of tangency.

It is now evident that the boundary of  $W(A)$  is a subset of  $C(A)$ , since for each  $\theta$ ,  $\xi = \xi_{\theta 1}$  is a member of the family of lines of which  $C(A)$  is the envelope. The equation presently obtained for  $C(A)$  is, therefore, the equation of the boundary of  $W(A)$  in the sense that the boundary of  $W(A)$  is identical with the convex outer portion of  $C(A)$ . The case  $m = 2$  is exceptional, in that  $C(A)$  is identical with the boundary of  $W(A)$ .

Expansion of the determinant given above as the characteristic equation of  $F_\theta$  yields a polynomial of degree  $m$  in  $\xi$ :

$$\sum_{r=0}^m \left[ (\xi^{m-r}/2^r) \sum_{j=0}^r p_{rj} \bar{t}^{r-2j} \right] = 0 \quad (1)$$

The  $p_{rj}$  are functions of the elements of  $A$ ; in particular,  $p_{00} = 1$ , and  $p_{r-r-j} = \bar{p}_{rj}$  for  $j = 0, 1, \dots, r$ , and  $r = 0, 1, \dots, m$ . Also, as a consequence of the fact that  $\sum_{i=1}^m \lambda_i = 0$ , note that  $p_{10} = p_{11} = 0$ . Finally, the

$m$  numbers  $(-1)^r \cdot p_{r0}$ , for  $r = 1, 2, \dots, m$ , are the symmetric functions of the  $\lambda_i$ . The other  $p_{rj}$ , which are, in general, rather complicated expressions in the  $\lambda_i$  and  $a_{ij}$ , have been evaluated only in the cases  $m = 2, 3, 4$ .

Let  $z$  be any point on any of the  $m$  lines  $\xi = \xi_{0j}$ ; then  $\xi = (z\bar{t} + \bar{z}t)/2$ . In (1), replace  $\xi$  by this value, and multiply by  $2^m$ :

$$\sum_{r=0}^m \left[ (z\bar{t} + \bar{z}t)^{m-r} \sum_{j=0}^r p_{rj} \bar{t}^{r-2j} \right] = 0 \quad (2)$$

Now consider a single term (that is, for a particular  $r$ ) of the above summation:

$$(z\bar{t} + \bar{z}t)^{m-r} \sum_{j=0}^r p_{rj} \bar{t}^{r-2j}$$

Multiply by  $\bar{t}^m$ , as follows:

$$\bar{t}^{m-r} (z\bar{t} + \bar{z}t)^{m-r} \bar{t}^r \sum_{j=0}^r p_{rj} \bar{t}^{r-2j}$$

which yields:

$$(z\bar{t}^2 + \bar{z})^{m-r} \sum_{j=0}^r p_{rj} \bar{t}^{2r-2j}$$

Now substitute  $s = \bar{t}^2$ :

$$(zs + \bar{z})^{m-r} \sum_{j=0}^r p_{rj} s^{r-j}$$

If this is done for each  $r$ , (2) becomes:

$$\sum_{r=0}^m \left[ (zs + \bar{z})^{m-r} \sum_{j=0}^r p_{rj} s^{r-j} \right] = 0 \quad (3)$$

The binomial expansion of  $(zs + \bar{z})^{m-r}$  may be written:

$$({}_{m-r}C_i) z^{m-r-i} s^{m-r-i} \bar{z}^i$$

On substituting this into (3) there is obtained:

$$\sum_{r=0}^m \left[ \sum_{i=0}^{m-r} ({}_{m-r}C_i) z^{m-r-i} \bar{z}^i s^{m-r-i} \sum_{j=0}^r p_{rj} s^{r-j} \right] = 0 \quad (4)$$

Now multiply the two polynomials in  $s$ :

$$\sum_{r=0}^m \left\{ \sum_{k=0}^m \left[ s^{m-k} \sum_{i+j=k} p_{ri} ({}_{m-r}C_j) z^{m-r-j} \bar{z}^j \right] \right\} = 0$$

The terms in  $s$  may be collected by interchanging the initial summation symbols (possible since these are finite sums):

$$\sum_{k=0}^m \sum_{r=0}^m \left[ s^{m-k} \sum_{i+j=k} p_{ri} ({}_{m-r}C_j) z^{m-r-j} \bar{z}^j \right] = 0$$

Since  $s^{m-k}$  is independent of  $r$ ,

$$\sum_{k=0}^m s^{m-k} \sum_{r=0}^m \sum_{i+j=k} p_{ri} ({}_{m-r}C_j) z^{m-r-j} \bar{z}^j = 0 \quad (5)$$

Regarding (5) as a polynomial in  $s$ , write:

$$\sum_{k=0}^m u_k s^{m-k} = 0 \quad (6)$$

where

$$u_k = \sum_{r=0}^m \sum_{i+j=k} p_{ri} ({}_{m-r}C_j) z^{m-r-j} \bar{z}^j$$

Two special properties of the  $u_k$  should be noted. Since

$({}_{m-r}C_j) = ({}_{m-r}C_{m-r-j})$ , and  $p_{r, r-j} = \bar{p}_{rj}$ , it is evident

that  $u_j = \bar{u}_{m-j}$ . Again, since the numbers  $(-1)^x p_{x_0}$  are the symmetric functions of the  $\lambda_i$ , and  $(\sum_{m-r} C_0) = 1$ ,  $u_0$  is the characteristic function of A.

Now  $z$  is any point on any of the  $m$  lines for an arbitrary  $\theta$ , so that (6) is the equation of the family of lines thus produced as  $\theta$  varies from 0 to  $2\pi$ . The envelope of the family is obtained by eliminating  $s$  between (6) and its derivative with respect to  $s$ ; that is, simply by setting the discriminant of (6) equal to zero. To show this, let (6) be denoted by  $f(s) = 0$ . Now  $df/d\theta = (df/ds)(ds/d\theta)$ . But  $s = \bar{t}^2$ ,  $ds/d\theta = -2i\bar{t}$ . Therefore,  $ds/d\theta$  is not zero for any value of  $\theta$ , and  $df/d\theta = 0$  is equivalent to  $df/ds = 0$ . That is, eliminating  $s$  from (6) is equivalent to eliminating  $\theta$ .

The discriminant of  $\sum_{\kappa=0}^m u_{\kappa} s^{m-\kappa}$  is the eliminant of this polynomial with its derivative (except for a constant factor). Thus the equation of  $C(A)$  in determinant form is:

$$\begin{vmatrix}
 u_0 & u_1 & \dots & u_{m-2} & u_{m-1} & u_m & 0 & \dots & 0 \\
 0 & u_0 & \dots & u_{m-3} & u_{m-2} & u_{m-1} & u_m & \dots & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & \dots & u_0 & u_1 & u_2 & u_3 & \dots & u_m \\
 mu_0 & (m-1)u_1 & \dots & 2u_{m-2} & u_{m-1} & 0 & 0 & \dots & 0 \\
 0 & mu_0 & \dots & 3u_{m-3} & 2u_{m-2} & u_{m-1} & 0 & \dots & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 0 & 0 & \dots & 0 & mu_0 & (m-1)u_1 & (m-2)u_2 & \dots & u_{m-1}
 \end{vmatrix} = 0$$

4. The Plücker characteristics of  $C(A)$ . Let  $n$  be the degree,  $m$  the class,  $\delta$  the number of nodes,  $\kappa$  the number of cusps,  $\tau$  the number of bitangents,  $\iota$  the number of inflections of a curve. These six numbers are called the Plücker numbers of a curve and are connected by the equations:

$$(1) \quad m = n(n-1) - 2\delta - 3\kappa$$

$$(11) \quad n = m(m-1) - 2\tau - 3\iota$$

$$(111) \quad \iota = 3n(n-2) - 6\delta - 8\kappa$$

Six others may be deduced from these, but are not listed, since they will not be needed in the discussion.<sup>3</sup>

It has been established earlier that if  $C(A)$  has a bitangent  $ax + by + c = 0$ , then  $(ax + by + c)^2$  is a factor of the equation of  $C(A)$ . Now assume that these linear factors have been dropped; then, except for normal matrices,

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<sup>3</sup> Ibid., p. 112.

there is a remaining factor, which will be denoted by  $C'(A)$ . Let  $N$  denote the degree of  $C(A)$ , and  $n$  the degree of  $C'(A)$ .

The curve  $C(A)$  has no inflection points, so that  $l = 0$ . The argument for this is based on the characteristic equation of  $F_\theta$ . The coefficient of  $\xi^x$  can be written  $\sum_i [b_{xi} \cos(m - r - 2i)\theta + c_{xi} \sin(m - r - 2i)\theta]$ , where  $i$  ranges from 0 to the integral part of  $(m - r)/2$ . The continuity of these coefficients, and the fact that all roots of  $F_\theta$  are real, imply that  $\xi$  is a continuous function of  $\theta$ . But  $\xi$  would be discontinuous at an inflection point. The fact that  $l = 0$  may be used to determine  $N$ .

**Theorem 5.3.** If  $A$  is an  $m$ -th order matrix, then the degree,  $N$ , of  $C(A)$  is  $m(m - 1)$ .

It has been shown that  $N = n + 2\tau$ . From Plücker's equation (11),  $n = m(m - 1) - 2\tau$ ; therefore,  $N = m(m - 1)$ .

It is possible for  $C(A)$  to be reducible in another way (that is, other than into  $C'(A)$  and linear factors representing bitangents): it may be factorable as  $f(x,y)g(x,y) = 0$ , where  $f(x,y) = 0$  and  $g(x,y) = 0$  are curves of classes necessarily less than  $m$ . If  $r$  is the class of the first factor its degree is  $r(r - 1)$ ; then the class of the remaining factor is  $m - r$ , and its degree  $(m - r)(m - r - 1)$ . Now two such curves will have  $r(m - r)$

common tangents (real or imaginary), and these common tangents are bitangents of  $C(A)$ , represented in the equation of  $C(A)$  by linear factors of total degree  $2r(m - r)$ .

Thus, the total degree of  $C(A)$  is

$$N = r(r - 1) + (m - r)(m - r - 1) + 2r(m - r) = m(m - 1).$$

An induction may be established to prove the theorem for more than two factors.

No attempt is made to discuss the very complicated problem of what happens to the Plücker characteristics in this type of reducibility. In the remainder of the present section,  $C'(A)$  is assumed to be irreducible, and therefore of class  $m$ .

Additional information on the Plücker characteristics of  $C(A)$  is obtained from Klein's equation.<sup>4</sup> Let  $k$  be the number of real cusps,  $d$  the number of real acnodes, or isolated points,  $i$  the number of real inflections, and  $t$  the number of ideal bitangents. The Klein's equation is  $n + i + 2t = m + k + 2d$ . Since  $i = 0$ , certainly  $i = 0$ . Again, since all real bitangents have real points of contact, there are no ideal bitangents. Hence, Klein's equation becomes  $k + 2d = n - m$ .

A comparison of Klein's equation with Plücker's equation (iv) yields a strong restriction on  $k$ .

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<sup>4</sup> Ibid., p. 361.



$$(iv) \quad K = 3m(m-2) - 6\tau - 8\iota.$$

Since  $\iota = 0$ ,  $K = 3m(m-2) - 6\tau$ . Now,  $n = m(m-1) - 2\tau$ , so that Klein's equation may be written

$$k + 2d = m(m-2) - 2\tau. \quad \text{Since } d \geq 0, \quad k \leq m(m-2) - 2\tau.$$

Thus, it is established that the number of real cusps is less than or equal to one-third of the total number of cusps; the equality holds if  $n = N$ . A proof of a theorem which is the dual of this with reference to the polar reciprocal curve is given by Coolidge.<sup>5</sup>

5. Murnaghan's equation. The article by Murnaghan referred to earlier<sup>6</sup> makes a different use of the characteristic equation of  $F_\theta$ . In the characteristic determinant of  $F_\theta$ , the diagonal element  $\xi = (\lambda_i \bar{t} + \bar{\lambda}_i t)/2$  is, for each  $k = 1, 2, \dots, m$ , the distance from  $\lambda_i$  to the line  $\xi = \xi_{\theta k}$ . Call this distance  $p_i$ . Then the equation may be written:

$$\begin{vmatrix} p_1 & (-a_{12} \bar{t})/2 & \dots & (-\bar{a}_{1m} t)/2 \\ (-\bar{a}_{12} t)/2 & p_2 & \dots & (-a_{2m} \bar{t})/2 \\ \dots & \dots & \dots & \dots \\ (-\bar{a}_{1m} t)/2 & (-\bar{a}_{2m} t)/2 & \dots & p_m \end{vmatrix} = 0$$

Murnaghan's principal conclusion is that the

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<sup>5</sup> J. L. Coolidge, Algebraic Plane Curves (Oxford: The Clarendon Press, 1931), p. 114.

<sup>6</sup> Murnaghan, op. cit., p. 248.

characteristic roots of  $A$  are in general the foci<sup>7</sup> of the curve here denoted by  $C(A)$ . For, in determining the foci, only the highest degree terms, here  $p_1 p_2 p_3 \dots p_m$ , occur; a tangent through a focus is of the form  $z = \text{a constant}$ , whereas, here, if  $z = \lambda_i$ , then  $p_i = 0$ , so that  $z = \lambda_i$  is necessarily a tangent through  $\lambda_i$ . Hence  $\lambda_i$  is a focus.

The relationships discussed here are used in following sections in the study of the special cases  $m = 2, 3, 4$ .

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<sup>7</sup> Hilton, op. cit., p. 69.

## VI

### The Second Order Matrix

1. A direct calculation for  $C(A)$ . Let  $A$  be the matrix

$$A = \begin{pmatrix} c & 2b \\ 0 & -c \end{pmatrix}$$

with  $b$  and  $c$  real. Then  $C(A)$  is the ellipse  $x^2/a^2 + y^2/b^2 = 1$ , where  $a^2 = b^2 + c^2$ . For the proof, the following lemma is useful:

Lemma 6.1. Let  $\lambda$  be a characteristic root of

$$B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

Then  $x B \bar{x}' = \lambda$ , if  $x = (1/N)(\bar{q}, \bar{\lambda} - \bar{p})$ , where

$$N = \sqrt{q\bar{q} + (\lambda - p)(\bar{\lambda} - \bar{p})}.$$

This vector is an obvious solution of  $(\lambda I - B)\bar{x}' = 0$ , and the conclusion of the lemma results from multiplying this equation on the left by  $x$ .

Theorem 6.1. The field of values of a second order matrix is an ellipse.

Let  $A$  be as given above; then the Hermitian decomposition of  $\bar{t}A$  is:

$$\bar{t}A = \begin{pmatrix} c \cos \theta & bt \\ bt & -c \cos \theta \end{pmatrix} + i \begin{pmatrix} -c \sin \theta & -ibt \\ ibt & c \sin \theta \end{pmatrix}$$

Application of the lemma to  $F_\theta$  yields a vector  $x$  for which  $x F_\theta \bar{x}' = \xi_{\theta 1}$  :  $x = (1/N)(bt, \xi_{\theta 1} - c \cos \theta)$ , where  $N = \sqrt{2 \xi_{\theta 1} (\xi_{\theta 1} - c \cos \theta)}$ . Then  $x(\bar{t}A)\bar{x}' = \xi_{\theta 1} + ix g_\theta \bar{x}'$ . Calculation of  $x g_\theta \bar{x}'$  yields  $(-c^2 \sin \theta \cos \theta / \xi_{\theta 1})$ . Now take the real and imaginary parts of the complex number  $x(\bar{t}A)\bar{x}'$  as Cartesian coordinates in the  $\xi\eta$  plane. The resulting point

$$\begin{aligned}\xi &= \xi_{\theta 1} \\ \eta &= (-c^2 \sin \theta \cos \theta) / \xi_{\theta 1}\end{aligned}$$

is a point of  $C(A)$  and, of course, also of the boundary of  $W(A)$ . Now, recalling that

$$\begin{aligned}x &= \xi \cos \theta - \eta \sin \theta \\ y &= \xi \sin \theta + \eta \cos \theta\end{aligned}$$

and calculating  $\xi_{\theta 1} = \sqrt{b^2 + c^2 \cos^2 \theta}$ , there is obtained

$$\begin{aligned}x &= \frac{(b^2 + c^2) \cos \theta}{\sqrt{b^2 + c^2 \cos^2 \theta}} \\ y &= \frac{b^2 \sin \theta}{\sqrt{b^2 + c^2 \cos^2 \theta}}\end{aligned}$$

On eliminating  $\theta$  and replacing  $b^2 + c^2$  by  $a^2$ , the equation  $x^2/a^2 + y^2/b^2 = 1$  is obtained.

This direct calculation, complicated for the second order matrix, is not practicable for higher orders.

For the general case, let  $A$  be the matrix

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

Then a unitary matrix  $U$  such that  $UA\bar{U}'$  is triangular is:

$$U = \begin{pmatrix} \bar{q}/N & (\bar{\lambda}_1 - \bar{p})/N \\ -(\lambda_1 - p)/N & q/N \end{pmatrix}$$

where  $N = \sqrt{q\bar{q} + (\lambda_1 - p)(\bar{\lambda}_1 - \bar{p})}$ . Calculation of  $UA\bar{U}'$  yields

$$B = UA\bar{U}' = \begin{pmatrix} \lambda_1 & \frac{(\bar{\lambda}_1 - \bar{p})[r(\bar{p} - \bar{s}) - \bar{q}(p - s)] + \bar{q}(q\bar{q} - r\bar{r})}{N} \\ 0 & \lambda_2 \end{pmatrix}$$

Let  $C = B - cI$ , where  $c = \frac{1}{2} \sum_{i=1}^2 \lambda_i$ ; then the diagonal elements of  $C$  are symmetrically placed with respect to the origin. Let the amplitudes be  $\alpha$  and  $-\alpha$ ; then the diagonal elements of  $D = e^{-i\alpha} C$  are real, so that  $D$  is of the form

$$D = \begin{pmatrix} c & 2be^{i\beta} \\ 0 & -c \end{pmatrix}$$

Now let  $V$  be the unitary matrix

$$V = \begin{pmatrix} e^{-i\beta} & 0 \\ 0 & 1 \end{pmatrix}$$

Then

$$E = VD\bar{V}' = \begin{pmatrix} c & 2b \\ 0 & -c \end{pmatrix}$$

$C(E)$ , as shown above, is  $x^2/a^2 + y^2/b^2 = 1$ ;  $C(A)$  can be found by performing on  $C(E)$  the inverses of the rotation and translation performed on  $D$  and  $C$  respectively.

Two special unitary transforms of the standard form  $E$  above are exhibited here as a matter of interest:

$$\begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \quad \begin{pmatrix} 0 & a+b \\ a-b & 0 \end{pmatrix}$$

## 2. The Plücker relations and Klein's equation.

According to the general discussion of section V, the degree of  $C(A)$  is  $m(m-1)$ . For  $m=2$ , therefore, the degree is 2. Since  $m=n=2$ , the Plücker relations imply that  $\delta = \kappa = \tau = \iota = 0$ . Klein's equation yields no additional information. These facts are, of course, consistent with the ellipse, now known to be  $C(A)$  for the second order case.

3. On finding  $C(A)$  by means of the discriminant of  $F_\theta$ . Let  $A$  be the second order matrix

$$A = \begin{pmatrix} c & 2b \\ 0 & -c \end{pmatrix}$$

For  $\bar{t}A = F_\theta + iG_\theta$ , the matrix  $F_\theta$  is given by

$$F_\theta = \begin{pmatrix} c(t + \bar{t})/2 & b\bar{t} \\ bt & -c(t + \bar{t})/2 \end{pmatrix}$$

The characteristic equation of  $F_\theta$  :

$$\xi^2 - \frac{1}{2}c^2(t + \bar{t})^2 \xi - b^2 = 0$$

Substitute  $\xi = \frac{1}{2}(z\bar{t} + \bar{z}t)$ , multiply by  $\bar{t}^2$ , and replace  $\bar{t}^2$  by  $s$ , to obtain

$$u_0 s^2 + u_1 s + u_2 = 0,$$

where  $u_0 = z^2 - c^2$ ,  $u_1 = 2z\bar{z} - 4b^2 - 2c^2$ ,  $u_2 = u_0$ .

The desired equation of  $C(A)$  is obtained by setting the discriminant of this polynomial in  $s$  equal to zero:

$$u_1^2 - 4u_0\bar{u}_0 = 0$$

$$\text{or } (2xz - 4b^2 - 2c^2)^2 - 4(z^2 - c^2)(\bar{z}^2 - c^2) = 0$$

On replacing  $z$  by  $x + iy$ , performing indicated operations, and combining similar terms, there is obtained, as before,  $x^2/a^2 + y^2/b^2 = 1$ .

4. Murnaghan's equation. In Murnaghan's notation the characteristic equation of  $F_0$  is:

$$\begin{vmatrix} p_1 & -b\bar{t} \\ -bt & p_2 \end{vmatrix} = 0$$

or on expanding,  $p_1 p_2 - b^2 = 0$ . Murnaghan points out that this is simply the algebraic formulation of the well known property of the ellipse that the product of the distances from the foci to a tangent is equal to the square of the minor axis.<sup>1</sup>

The corresponding statement for the hyperbola is  $p_1 p_2 + b^2 = 0$ . Thus, the equation  $p_1 p_2 = k$  yields an ellipse or an hyperbola according as  $k$  is positive or negative, and a straight line segment if  $k = 0$ . This situation may be generalized, since evidently any class  $m$  curve with  $m$  real foci may be expressed in the form of a polynomial equation of degree  $m$  in the  $p_i$ . One variety of curve that results is unbounded, and related to the bounded variety which is  $G(A)$  for some matrix  $A$  in a manner

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<sup>1</sup> H. B. Fine and H. D. Thompson, Coordinate Geometry (New York: The Macmillan Company, 1927), p. 85.

analogous to the relationship of the hyperbola to the ellipse.

5. A family of Cassini ovals associated with a second order matrix. In a recent paper,<sup>2</sup> Brauer proved that, if  $p_k = \sum_{\substack{j=1 \\ j \neq k}}^m |a_{kj}|$ , for  $k = 1, 2, \dots, m$ , then each characteristic root,  $\lambda$ , of a matrix  $A = (a_{ij})$  lies in the interior or on the boundary of at least one of the  $m(m-1)/2$  Cassini ovals

$$|z - a_{ii}| |z - a_{kk}| = p_i p_k$$

In case  $n = 2$ , there is only one Cassini oval and the characteristic roots are on its boundary. Let  $M$  be the set of matrices which are unitary transforms of a given second order matrix  $A$ . For each matrix of  $M$  there is a unique Cassini oval, so that there is determined a set,  $C$ , of Cassini ovals associated with  $A$  and hence with  $W(A)$ , the fixed ellipse which is the field of values of each member of  $M$ .

The foci of a given Cassini oval of  $C$  are the principal diagonal elements of the corresponding matrix of  $M$ . Now the principal diagonal elements of a matrix belong to its field of values, and by Theorem 2.2, have the same centroid as the characteristic roots. Therefore, the foci of any Cassini oval of  $C$  are a pair of points within or on

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<sup>2</sup> Alfred Brauer, "Limits for the Characteristic Roots of a Matrix," Duke Mathematical Journal, XIV, (1947), 21-26.



the ellipse and symmetrically placed with respect to its center. Conversely, each two such points,  $p$  and  $q$ , are the foci of some Cassini oval of  $C$ , for, to obtain the corresponding matrix of  $M$ , it suffices to take as the first row of  $U$  a unit vector  $u$  such that  $uAu' = p$ .

Though for a given matrix  $A$ , Brauer's theorem restricts the possibilities for the characteristic roots to the boundary of a single Cassini oval, the totality of all points which could arise as possibilities in this way evidently consists of the set of all boundary points on all of the Cassini ovals of  $C$ , since the given matrix  $A$  may have been any arbitrary member of  $M$ . A comparison of the resulting set of points is obtained in the following theorem.

Theorem 6.2. The set of points lying on the boundaries of the Cassini ovals of  $C$  consists of the interior and circumference of the circle concentric with the ellipse and with radius  $\sqrt{a^2 + b^2}$ .

The ellipse may be assumed in standard position. Evidently the Cassini ovals which extend farthest from the center are those whose foci are on the circumference of the ellipse. Consider, then, the subfamily  $C'$  of Cassini ovals whose foci are at  $(u,v)$  and  $(-u,-v)$  for  $u^2/a^2 + v^2/b^2 = 1$ . Let  $r$  and  $s$  be the constants of the Cassini oval; thus  $r$  is the distance from the center to a

focus, and  $s^2$  is the product of the distances from a point on the oval to the two foci. Then

$$\begin{aligned} r^2 &= u^2 + v^2 \\ &= (a^2 b^2 + c^2 u^2)/a^2 \end{aligned}$$

And since the oval passes through the foci of the ellipse,

$$\begin{aligned} s^2 &= \left[ (c - u)^2 + v^2 \right]^{\frac{1}{2}} \left[ (c + u)^2 + v^2 \right]^{\frac{1}{2}} \\ &= (a^4 - c^2 u^2)/a^2 \end{aligned}$$

The sum of these two quantities is

$$r^2 + s^2 = a^2 + b^2$$

But  $\sqrt{r^2 + s^2}$  is the distance from the origin to the end of the major axis of the Cassini oval, so that this distance, for members of  $C'$ , is constantly  $\sqrt{a^2 + b^2}$ . Furthermore, all other points of the oval are closer to the origin. It is now evident that each point of the circle of the theorem is in the interior or on the boundary of some Cassini oval of  $C$ ; each point in or on the ellipse is a focus for some oval of  $C$ , and, therefore, certainly interior to it, and each point outside the ellipse but inside or on the circle lies on the line joining the focus of some member of  $C'$  to the end of its major axis.

To complete the proof, it is possible by considerations of continuity to show that each point of the circle is actually on the boundary of some oval. The equation of the family of Cassini ovals can be written parametrically

$$\begin{aligned}
& \left[ (x - at \cos \theta)^2 + (y - bt \sin \theta)^2 \right] \left[ (x + at \cos \theta)^2 \right. \\
& \quad \left. + (y + bt \sin \theta)^2 \right] \\
& = \left[ (c - at \cos \theta)^2 + (-bt \sin \theta)^2 \right] \left[ (c + at \cos \theta)^2 \right. \\
& \quad \left. + (bt \sin \theta)^2 \right]
\end{aligned}$$

for the ranges  $0 \leq t \leq 1$ , and  $0 \leq \theta \leq 2\pi$ . Representing this equation by the number pair  $(t, \theta)$ , it is evident that the points  $(t, \theta)$  in the  $t, \theta$  plane fill up the interior and boundary of a rectangle. For a given point  $p$  in the circle of the theorem, let  $(t_0, \theta_0)$  represent a Cassini oval containing  $p$  in its interior or on its boundary. Now the Cassini oval corresponding to the number pair  $(0, e)$ , where  $e$  is the eccentricity of the ellipse, is a degenerate case consisting simply of the two foci of the ellipse. Thus, there is exhibited a Cassini oval containing  $p$ , and one not containing  $p$ . The corresponding points  $(t_0, \theta_0)$  and  $(0, e)$  in the  $t, \theta$  plane determine a straight line segment, each point of which corresponds to a Cassini oval of  $C$ ; then some point of the segment corresponds to an oval passing through  $p$ .

The diagonals of the circumscribing rectangle of the ellipse with sides parallel to the axes of the ellipse are diameters of the circle of the theorem and serve to classify the ovals of  $C'$  into those with two loops, the lemniscates, those with one loop, according as the foci are in the smaller angle formed by the diagonals, on the diagon-

al, or in the larger angle.

Some special cases are of interest. For the matrix

$$A = \begin{pmatrix} 0 & a + b \\ a - b & 0 \end{pmatrix}$$

the associated Cassini oval degenerates into two coincident circles, with the line joining the foci of the ellipse as a diameter. For the case  $b = 0$ , the ellipse degenerates to a straight line segment, and the circle of the theorem has this line segment as a diameter. For the case  $c = 0$ , the ellipse degenerates to a circle, and the associated Cassini ovals are all lemniscates.

## VII

### The Third Order Matrix

1. Calculation of the equation of  $C(A)$ . The method used in this calculation is the one described in V, involving the use of the discriminant of a certain polynomial in  $s$ . Let  $A$  be the matrix

$$A = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} \\ 0 & \lambda_2 & a_{23} \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

where  $\sum_{i=1}^3 \lambda_i = 0$ . Then, if  $\bar{t}A = F_\theta + iG_\theta$ ,

$$F_\theta = \begin{pmatrix} (\lambda_1 \bar{t} + \bar{\lambda}_1 t)/2 & (a_{12} \bar{t})/2 & (a_{13} \bar{t})/2 \\ (\bar{a}_{12} t)/2 & (\lambda_2 \bar{t} + \bar{\lambda}_2 t)/2 & (a_{23} \bar{t})/2 \\ (\bar{a}_{13} t)/2 & (\bar{a}_{23} t)/2 & (\lambda_3 \bar{t} + \bar{\lambda}_3 t)/2 \end{pmatrix}$$

The characteristic equation of  $F_\theta$  is:

$$\xi^3 + \frac{1}{t} \xi^2 (p_{20} \bar{t}^2 + p_{21} + p_{22} t^2) + \frac{1}{t^2} (p_{30} \bar{t}^3 + p_{31} \bar{t} + p_{32} t + p_{33} t^3) = 0$$

where  $p_{20} = \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3$

$$p_{21} = -(\lambda_1 \bar{\lambda}_1 + \lambda_2 \bar{\lambda}_2 + \lambda_3 \bar{\lambda}_3 + a_{12} \bar{a}_{12} + a_{13} \bar{a}_{13} + a_{23} \bar{a}_{23})$$

$$p_{22} = \bar{p}_{20}$$

$$p_{30} = -\lambda_1 \lambda_2 \lambda_3$$

$$p_{31} = -(\lambda_1 \lambda_2 \bar{\lambda}_3 + \lambda_1 \bar{\lambda}_2 \lambda_3 + \bar{\lambda}_1 \lambda_2 \lambda_3)$$

$$+ a_{12} \bar{a}_{12} \lambda_3 + a_{13} \bar{a}_{13} \lambda_2 + a_{13} \bar{a}_{23} \lambda_1 - a_{12} \bar{a}_{13} a_{23}$$

$$p_{32} = \bar{p}_{31}$$

$$p_{33} = \bar{p}_{30}$$

In this equation, substitute  $\xi = \frac{1}{2}(z\bar{t} + \bar{z}t)$ , multiply by  $(2\bar{t})^3$ , and let  $\bar{t}^2 = s$ :

$$(zs + \bar{z})^3 + (zs + \bar{z})(p_{20}s^2 + p_{21}s + \bar{p}_{20}) + (p_{30}s^3 + p_{31}s^2 + \bar{p}_{31}s + \bar{p}_{30}) = 0$$

Perform indicated operations and collect terms in  $s$ , to obtain:

$$u_0 s^3 + u_1 s^2 + u_2 s + u_3 = 0$$

where  $u_0 = z^3 + p_{20}z + p_{30}$

$$u_1 = 3z^2\bar{z} + p_{21}z + p_{20}\bar{z} + p_{31}$$

$$u_2 = \bar{u}_1$$

$$u_3 = \bar{u}_0$$

Then  $u_0 s^3 + u_1 s^2 + \bar{u}_1 s + \bar{u}_0 = 0$ . Now, according to V, the equation sought is simply the discriminant of this polynomial in  $s$  set equal to zero;<sup>1</sup>

$$u_1^2 \bar{u}_1^2 - 4u_0 \bar{u}_1^3 - 4\bar{u}_0 u_1^3 + 18u_0 \bar{u}_0 u_1 \bar{u}_1 - 27 u_0^2 \bar{u}_0^2 = 0.$$

Replacing the  $u_i$  by their values as given above yields the equation of  $C(A)$  in terms of the complex variable  $z$ , from which the Cartesian equation follows on substituting

$z = x + iy$ . The equation in terms of  $z$ :

$$\begin{aligned} & z^6 (-4\bar{p}_{20}^3 - 27\bar{p}_{30}^2) + z^5 \bar{z} (54\bar{p}_{30} \bar{p}_{31} + 12\bar{p}_{20}^2 p_{21}) \\ & + z^4 \bar{z}^2 (-27\bar{p}_{31}^2 - 12\bar{p}_{20} p_{21}^2 - 12p_{20} \bar{p}_{20}^2 - 54\bar{p}_{30} p_{31}) \\ & + z^3 \bar{z}^3 (4p_{21}^3 + 54p_{31} p_{31} + 24p_{20} \bar{p}_{20} p_{21} + 54p_{30} \bar{p}_{30}) \end{aligned}$$

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<sup>1</sup> W. S. Burnside and A. W. Patton, The Theory of Equations (Seventh Edition; Dublin: Dublin University Press, 1912), I, p. 83.

$$\begin{aligned}
& + z^2 \bar{z}^4 (-27p_{31}^2 - 12p_{20} p_{21}^2 - 12p_{20}^2 \bar{p}_{20} - 54p_{30} \bar{p}_{31}) \\
& + z \bar{z}^5 (54p_{30} p_{31} + 12p_{20}^2 p_{21}) + \bar{z}^6 (-4p_{20}^3 - 27p_{20}^2) \\
& + z^5 (-12\bar{p}_{20}^2 \bar{p}_{31} + 18\bar{p}_{20} p_{21} \bar{p}_{30}) \\
& + z^4 \bar{z} (6\bar{p}_{20} p_{21} \bar{p}_{31} + 24\bar{p}_{20}^2 p_{31} - 18p_{21}^2 \bar{p}_{30} - 36p_{20} \bar{p}_{20} \bar{p}_{30}) \\
& + z^3 \bar{z}^2 (6p_{21}^2 \bar{p}_{31} + 12p_{20} \bar{p}_{20} \bar{p}_{31} - 30\bar{p}_{20} p_{21} p_{31} \\
& \quad + 54p_{20} p_{21} \bar{p}_{30} - 36\bar{p}_{20}^2 p_{30}) \\
& + z^2 \bar{z}^3 (6p_{21}^2 p_{31} + 12p_{20} \bar{p}_{20} p_{31} - 30p_{20} p_{21} \bar{p}_{31} \\
& \quad + 54\bar{p}_{20} p_{21} p_{30} - 36p_{20}^2 \bar{p}_{30}) \\
& + z \bar{z}^4 (6p_{20} p_{21} p_{31} + 24p_{20}^2 \bar{p}_{31} - 18p_{21}^2 p_{30} - 36p_{20} \bar{p}_{20} p_{30}) \\
& + \bar{z}^5 (-12p_{20}^2 p_{31} + 18p_{20} p_{21} p_{30}) \\
& + z^4 (\bar{p}_{20}^2 p_{21}^2 - 12\bar{p}_{20} \bar{p}_{31}^2 - 4p_{20} \bar{p}_{20}^3 + 18p_{21} \bar{p}_{30} \bar{p}_{31} \\
& \quad + 18\bar{p}_{20} \bar{p}_{30} p_{31} - 54p_{20} \bar{p}_{30}^2) \\
& + z^3 \bar{z} (-6p_{21} \bar{p}_{31}^2 + 30\bar{p}_{20} p_{31} \bar{p}_{31} + 8p_{20} \bar{p}_{20}^2 p_{21} - 2\bar{p}_{20} p_{21}^3 \\
& \quad - 54p_{21} \bar{p}_{30} p_{31} + 72p_{20} \bar{p}_{30} p_{31} - 54\bar{p}_{20} p_{30} \bar{p}_{30}) \\
& + z^2 \bar{z}^2 (-30p_{20} \bar{p}_{31}^2 + 24p_{21} p_{31} \bar{p}_{31} - 2p_{20} \bar{p}_{20} p_{21}^2 \\
& \quad - 30\bar{p}_{20} p_{31}^2 + p_{21}^4 - 18p_{20} \bar{p}_{30} p_{31} - 18\bar{p}_{20} p_{30} p_{31} \\
& \quad + 108p_{21} p_{30} \bar{p}_{30} - 8p_{20}^2 \bar{p}_{20}^2) \\
& + z \bar{z}^3 (-6p_{21} p_{31}^2 + 30p_{20} p_{31} \bar{p}_{31} + 8p_{20}^2 \bar{p}_{20} p_{21} - 2p_{20} p_{21}^3 \\
& \quad - 54p_{21} p_{30} \bar{p}_{31} + 72\bar{p}_{20} p_{30} p_{31} - 54p_{20} p_{30} \bar{p}_{30}) \\
& + \bar{z}^4 (p_{20}^2 p_{21}^2 - 12p_{20} p_{31}^2 - 4p_{20}^3 \bar{p}_{20} + 18p_{20} p_{30} \bar{p}_{31} \\
& \quad + 18p_{21} p_{30} p_{31} - 54\bar{p}_{20} p_{30}^2) \\
& + z^3 (2\bar{p}_{20} p_{21}^2 \bar{p}_{31} + 2\bar{p}_{20}^2 p_{21} p_{31} - 4\bar{p}_{31}^3 - 12p_{20} \bar{p}_{20}^2 \bar{p}_{31} \\
& \quad - 4p_{21}^3 \bar{p}_{30} - 4\bar{p}_{20}^3 p_{30} + 18\bar{p}_{30} p_{31} \bar{p}_{31} \\
& \quad + 18p_{20} \bar{p}_{20} p_{21} \bar{p}_{30} - 54p_{30} \bar{p}_{30}^2)
\end{aligned}$$

$$\begin{aligned}
& + z^2 \bar{z} (6p_{21} \bar{p}_{31}^2 - 2p_{20} \bar{p}_{20} p_{21} \bar{p}_{31} + 2p_{21}^3 \bar{p}_{31} - 8\bar{p}_{20} p_{21}^2 p_{31} \\
& \quad + 20p_{20} \bar{p}_{20}^2 p_{31} - 36\bar{p}_{30} p_{31}^2 + 6p_{20} p_{21}^2 \bar{p}_{30} \\
& \quad + 6\bar{p}_{20}^2 p_{21} p_{30} - 36p_{20}^2 \bar{p}_{20} \bar{p}_{30} + 54p_{30} \bar{p}_{30} \bar{p}_{31} ) \\
& + z \bar{z}^2 (6p_{31}^2 \bar{p}_{31} - 2p_{20} \bar{p}_{20} p_{21} p_{31} + 2p_{21}^3 p_{31} - 8p_{20} p_{21}^2 \bar{p}_{31} \\
& \quad + 20p_{20}^2 \bar{p}_{20} \bar{p}_{31} - 36p_{30} \bar{p}_{31}^2 + 6\bar{p}_{20} p_{21}^2 p_{30} \\
& \quad + 6p_{20}^2 p_{21} \bar{p}_{30} - 36p_{20} \bar{p}_{20}^2 p_{30} + 54p_{30} \bar{p}_{30} p_{31} ) \\
& + \bar{z}^3 (2p_{20}^2 p_{21} \bar{p}_{31} + 2p_{20} p_{21}^2 p_{31} - 4p_{31}^3 - 12p_{20}^2 \bar{p}_{20} p_{31} \\
& \quad - 4p_{21}^3 p_{30} - 4p_{20}^3 \bar{p}_{30} + 18p_{30} p_{31} \bar{p}_{31} \\
& \quad + 18p_{20} \bar{p}_{20} p_{21} p_{30} - 54p_{30}^2 \bar{p}_{30} ) \\
& + z^2 (p_{21}^2 \bar{p}_{31}^2 + 4\bar{p}_{20} p_{21} p_{31} \bar{p}_{31} + \bar{p}_{20}^2 p_{31}^2 - 12p_{20} \bar{p}_{20} \bar{p}_{31}^2 \\
& \quad - 12p_{21}^2 \bar{p}_{30} p_{31} - 12\bar{p}_{20}^2 p_{30} \bar{p}_{31} + 18p_{20} p_{21} \bar{p}_{30} \bar{p}_{31} \\
& \quad + 18p_{20} \bar{p}_{20} \bar{p}_{30} p_{31} + 18\bar{p}_{20} p_{21} p_{30} \bar{p}_{30} - 27p_{20}^2 \bar{p}_{30}^2 ) \\
& + z \bar{z} (-10p_{20} p_{21} \bar{p}_{31}^2 - 10\bar{p}_{20} p_{21} p_{31}^2 + 4p_{21}^2 p_{31} \bar{p}_{31} \\
& \quad + 22p_{20} \bar{p}_{20} p_{31} \bar{p}_{31} - 6p_{20} p_{21} \bar{p}_{30} p_{31} - 6\bar{p}_{20} p_{21} p_{30} \bar{p}_{31} \\
& \quad + 18p_{20}^2 \bar{p}_{30} \bar{p}_{31} + 18\bar{p}_{20}^2 p_{30} p_{31} + 18p_{21}^2 p_{30} \bar{p}_{30} \\
& \quad - 90p_{20} \bar{p}_{20} p_{30} \bar{p}_{30} ) \\
& + \bar{z}^2 (p_{20}^2 \bar{p}_{31}^2 + 4p_{20} p_{21} p_{31} \bar{p}_{31} + p_{21}^2 p_{31}^2 - 12p_{20} \bar{p}_{20} p_{31}^2 \\
& \quad - 12p_{21}^2 p_{30} \bar{p}_{31} - 12\bar{p}_{20}^2 \bar{p}_{30} p_{31} + 18p_{20} \bar{p}_{20} p_{30} \bar{p}_{31} \\
& \quad + 18\bar{p}_{20} p_{21} p_{30} p_{31} + 18p_{20} p_{21} p_{30} \bar{p}_{30} - 27\bar{p}_{20}^2 p_{30}^2 ) \\
& + z (2p_{21} p_{31} \bar{p}_{31}^2 + 2\bar{p}_{20} p_{31}^2 \bar{p}_{31} - 4p_{20} \bar{p}_{31}^3 - 12p_{21} \bar{p}_{30} p_{31}^2 \\
& \quad - 12\bar{p}_{20} p_{30} \bar{p}_{31}^2 + 18p_{20} \bar{p}_{30} p_{31} \bar{p}_{31} + 18p_{21} p_{30} \bar{p}_{30} \bar{p}_{31} \\
& \quad + 18\bar{p}_{20} p_{30} \bar{p}_{30} p_{31} - 54p_{20} p_{30} \bar{p}_{30}^2 ) \\
& + \bar{z} (2p_{20} p_{31} \bar{p}_{31}^2 + 2p_{21} p_{31}^2 \bar{p}_{31} - 4\bar{p}_{20} p_{31}^3 - 12p_{21} p_{30} p_{31}^2 \\
& \quad - 12p_{20} \bar{p}_{30} p_{31}^2 + 18\bar{p}_{20} p_{30} p_{31} \bar{p}_{31} + 18p_{20} p_{30} \bar{p}_{30} \bar{p}_{31} \\
& \quad + 18p_{21} p_{30} \bar{p}_{30} p_{31} - 54\bar{p}_{20} p_{30}^2 \bar{p}_{30} ) \\
& + (p_{31}^2 \bar{p}_{31}^2 - 4p_{30} \bar{p}_{31}^3 - 4\bar{p}_{30} p_{31}^3 + 18p_{30} \bar{p}_{30} p_{31} \bar{p}_{31} \\
& \quad - 27p_{30}^2 \bar{p}_{30}^2 ) = 0
\end{aligned}$$



Note that the coefficient of  $\bar{z}^6$  is the discriminant of the characteristic equation of  $A$ . If this discriminant is zero, then evidently  $z\bar{z} = x^2 + y^2$  is a factor of the sixth degree terms of the equation of  $C(A)$ ; thus, it is established that if  $A$  has a multiple root, then  $C(A)$  passes through the circular points at infinity. Moreover, if  $A$  has a triple root, so that  $p_{20} = p_{30} = 0$ , then the sixth degree terms have  $z^2\bar{z}^2$  as a factor, and the circular points at infinity are double points of  $C(A)$ .

2. Plücker's formulas and Klein's equation. If  $A$  is a third order matrix, then the degree of  $C(A)$  is  $N = m(m - 1) = 6$ , a result borne out by the equation above. If this equation is irreducible, its Plücker numbers are as shown in row (a) of the tabulation below. As indicated, Klein's equation says that  $k + 2d = 3$ , which has two solutions in non-negative integers:  $k = 3, d = 0$ , and  $k = 1, d = 1$ . The second solution is excluded, since  $d \leq \delta \leq 0$ . Several of the examples given later, in particular Example 1, illustrate this type of curve and show how the three real cusps arise as the vertices of a "triangular" region in the interior of the convex boundary.

The reducible cases may be classified according as there are 1, 2, or 3 bitangents. Only in the case of 1 bitangent is the curve genuinely of class 3, and the Plücker numbers of this type are shown below in row (b). For the

other cases, the curve is of class 3 in a degenerate sense described below under (c) and (d); the Plucker numbers have no significance in these cases, so that they do not show in the tabulation.

	m	n	$\delta$	$\kappa$	$l$	$\tau$	$(k + 2d)$
(a)	3	6	0	9	0	0	3
(b)	3	4	0	3	0	1	1

The relationship between the reducible cases and the irreducible case (a) may be brought out by considering them as resulting from certain deformations of type (a).

(b) 1 bitangent. Let one side of the triangular or three-cusp branch be deformed into coincidence with the convex branch of the curve, thus forming a bitangent of  $C(A)$ , represented in the equation by the square of a linear factor. The remaining factor,  $C'(a)$ , is, therefore, a quartic, and belongs to the Plucker type which includes the cardioid. The Klein equation  $k + 2d = 1$  has only one solution in non-negative integers:  $k = 1$ ,  $d = 0$ . For illustrations, see Examples 7 and 8. The cardioid is actually obtained in Example 8.

(c) 2 bitangents. In this case,  $n = N - 2\tau - 4 = 2$ ; hence the remaining factor is an ellipse. Let two sides of the triangular branch be deformed into coincidence with portions of the convex branch; then the configuration is an ellipse with two tangents from an external point, the point

of intersection of the two bitangents. Again, let the triangular branch shrink down to a point; then the configuration is an ellipse plus a single interior point, from which there are two imaginary tangents to the ellipse.

In terms of the family of three lines which envelope  $C(A)$ , this case arises when one of the lines passes through a fixed point for each  $\theta$ ; thus a line through this fixed point must be regarded as "tangent" to  $C(A)$ . The two remaining lines of the family must then envelope a class 2 curve; and since this class 2 curve must be convex, it must be an ellipse. Now if one of the "tangents" through the fixed point is also tangent to the ellipse, it is a bitangent of  $C(A)$ ; hence the two bitangents described above. Thus the ellipse plus fixed point may properly be regarded as constituting a class 3 curve. The fixed point, incidentally, is one of the characteristic roots of  $A$ , a property clarified in paragraph 4, below: See Examples 9 and 10.

(d) 3 bitangents. Here,  $n = N - 2\tau = 0$ . Let all three sides of the triangular branch be deformed into coincidence with portions of the convex branch; since each bitangent is represented in the equation by the square of a linear factor, these linear factors add up to a degree of 6, hence constitute the entire equation of  $C(A)$ . The three points of intersection of the three bitangents are fixed points of the same kind as the one discussed under (c), and

these three fixed points constitute a class 3 curve in the sense agreed on there. This case, of course, arises from a normal matrix. See example 11.

This completes the classification of the types of curves obtained from third order matrices.

3. Murnaghan's equation. In Murnaghan's form, the characteristic equation of  $F_0$  may be written

$$p_1 p_2 p_3 - \frac{1}{4}(a_{13} \bar{a}_{23} p_1 + a_{13} \bar{a}_{13} p_2 + a_{12} \bar{a}_{12} p_3) \\ + \frac{1}{8}(a_{12} \bar{a}_{13} a_{23} \bar{t} + \bar{a}_{12} \bar{a}_{13} \bar{a}_{23} t) = 0$$

Murnaghan writes this more compactly by introducing an auxiliary point,  $\alpha$ , given by

$$\alpha = \frac{a_{13} \bar{a}_{23} \lambda_1 + a_{13} \bar{a}_{13} \lambda_2 + a_{12} \bar{a}_{12} \lambda_3 - a_{12} \bar{a}_{13} a_{23}}{a_{12} \bar{a}_{12} + a_{13} \bar{a}_{13} + a_{23} \bar{a}_{23}}$$

Let  $k = a_{12} \bar{a}_{12} + a_{13} \bar{a}_{13} + a_{23} \bar{a}_{23}$ . Then the equation above may be written  $p_1 p_2 p_3 = \frac{1}{4} k p_\alpha$ , where  $p_\alpha$  denotes the distance from the point  $\alpha$  to the tangent line  $\xi = \xi_{oj}$ . The factor  $\frac{1}{4}$  is missing in Murnaghan's article, presumably a typographical error.

4. On class 3 curves with three real foci. The equation  $p_1 p_2 p_3 = \frac{1}{4} k p_\alpha$  describes a property of all class 3 curves with three real foci.<sup>2</sup> The point here denoted by

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<sup>2</sup> H. Hilton and S. D. Jervis, "On the Real Foci and Directrices of a Class Cubic and other Plane Algebraic Curves," Proceedings of the London Mathematical Society, XXVII (1928), p. 430.

is called the pole by Hilton. According to Hilton, the pole has the interesting property that it is the point of concurrency of the real tangents from the foci. Since the curve is of class 3, there are three tangents to the curve from a focus,  $\lambda_i$ . Two of them are the imaginary tangents through the circular points at infinity, according to the definition of a focus, but the third is real, and the three real tangents thus determined by the three real foci are concurrent at  $\alpha$ , since  $p_i = 0$  implies  $p_\alpha = 0$ , for  $i = 1, 2, 3$ .

It is now possible to clarify the situation discussed above under (e). If the point  $\alpha$  coincides with one of the  $\lambda_i$ , say  $\lambda_3$ , then the p-equation may be factored  $p_3(p_1 p_2 - \frac{1}{4}k) = 0$ , so that the resulting curve is one of the two types under (c), according as the point  $\alpha$  is inside or outside the ellipse determined by  $p_1 p_2 - \frac{1}{4}k = 0$ .

Type (d) is given by  $k = 0$ ; in this case,  $\alpha$  is indeterminate.

It is clear that a matrix A determines uniquely a set of four points,  $\lambda_1, \lambda_2, \lambda_3, \alpha$ , and a number k, and that these determine uniquely the curve  $C(A)$ , according to the equation  $p_1 p_2 p_3 - \frac{1}{4}kp_\alpha = 0$ . But, on the other hand, it may be observed that  $C(A)$  may also be  $C(B)$ , for some matrix B which is not a unitary transform of A. To illustrate, let A and B be as follows:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \omega & -3 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & \sqrt{11} \\ 0 & 0 & \omega^2 \end{pmatrix}$$

In each case, the  $\lambda_i$  are the cube roots of 1,  $k = 11$ ,  $\alpha = 1$ . That A and B are not unitary transforms follows from Theorem 3.3.

A statement analogous to the statement made above for matrices may be made for class 3 curves with real foci. Each class 3 curve with three real foci determines uniquely a set of four points,  $\lambda_1, \lambda_2, \lambda_3$ , and  $\alpha$ , and a real number  $k$  such that the equation of the curve may be expressed  $p_1 p_2 p_3 = \frac{1}{k} k p_\alpha$ , according to the theorem of Hilton already referred to. Thus, it becomes evident that the equation in  $z$  given for  $C(A)$  is perfectly general, including all class 3 curves with three real foci, for the  $p_{\lambda_i}$  may be expressed in terms of  $k$  and  $\alpha$  and the  $\lambda_i$  as follows:

$$\begin{aligned} p_{10} &= \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 \\ p_{21} &= -(\lambda_1 \bar{\lambda}_1 + \lambda_2 \bar{\lambda}_2 + \lambda_3 \bar{\lambda}_3 + k) \\ p_{30} &= -\lambda_1 \lambda_2 \lambda_3 \\ p_{31} &= k\alpha - (\lambda_1 \lambda_2 \bar{\lambda}_3 + \lambda_1 \bar{\lambda}_2 \lambda_3 + \bar{\lambda}_1 \lambda_2 \lambda_3). \end{aligned}$$

That the resulting curve need not be associated with a matrix A as  $C(A)$  has already been pointed out, but may now be established in a new way. For a given class 3 curve,  $k$  and  $\alpha$  are known; consider the equations defining them in terms of

the  $a_{ij}$  as a system in the unknowns  $a_{12}$ ,  $a_{13}$ , and  $a_{23}$ .

The system is not always solvable; for example, it is not solvable for  $\lambda_1 = 1$ ,  $\lambda_2 = 0$ ,  $\lambda_3 = -1$ ,  $\alpha = -2$ ,  $k = 1$ . For the class 3 curve with these values, see Example 12.

5. Numerical examples. Example 7.1.

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 0 & -2 + 1 & 1 - 1 \\ 0 & 0 & -1 - 1 \end{pmatrix}$$

$k = 4$ ,  $\alpha = (2 + 1)/4$ . The Cartesian equation is:

$$\begin{aligned} &1940x^6 + 6096x^5y + 7320x^4y^2 - 13184x^3y^3 + 157908x^2y^4 \\ &+ 101424xy^5 + 78800y^6 - 11328x^5 - 29088x^4y + 77568x^3y^2 \\ &+ 20064x^2y^3 + 561600xy^4 + 185856y^5 + 7008x^4 - 23256x^3y \\ &+ 9168x^2y^2 - 30456xy^3 + 192336y^4 + 64432x^3 + 133692x^2y \\ &- 770496xy^2 - 265876y^3 - 107696x^2 - 95552xy - 560060y^2 \\ &- 29804x + 69608y + 79607 = 0. \end{aligned}$$

Example 7.2.

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

$k = 3$ ,  $\alpha = -1/3$ . The Cartesian equation of  $C(A)$  is:

$$\begin{aligned} &4752y^6 + y^4(6156x^2 + 9180x + 567) + y^2(1512x^4 + 2376x^3 \\ &- 3114x^2 - 7236x - 3042) + (108x^6 + 108x^5 - 369x^4 \\ &- 644x^3 - 378x^2 - 96x - 9) = 0. \end{aligned}$$

**Example 7.3.**

$$A = \begin{pmatrix} 1 & 3/2 & 3/2 \\ 0 & \omega & 3 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$k = 27/2$ ,  $\alpha = 0$ , the centroid of the foci, which are the cube roots of unity. Cartesian equation of  $C(A)$ :

$$10648x^6 + 31368x^4y^2 + 32328x^2y^4 + 10584y^6 - 5808x^5$$

$$+ 11616x^3y^2 + 17424xy^4 - 42867x^4 - 85734x^2y^2 - 42867y^4$$

$$+ 21232x^3 - 63696xy^2 - 2904x^2 - 2904y^2 + 16 = 0.$$
**Example 7.4.**

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & \omega & 1 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$k = 3$ ,  $\alpha = -1/3$ . The Cartesian equation of  $C(A)$ :

$$864x^6 + 2160x^4y^2 + 3456x^2y^4 + 432y^6 - 864x^5$$

$$+ 3456x^3y^2 + 4320xy^4 - 1008x^4 - 2340x^2y^2 + 432y^4$$

$$+ 832x^3 - 5760xy^2 - 1584y^2 - 96x + 16 = 0.$$
**Example 7.5.**

$$A = \begin{pmatrix} 1 & 3/2 & 3/2 \\ 0 & \omega & 3/2 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$k = 27/4$ ,  $\alpha = -\frac{1}{2}$ . The Cartesian equation of  $C(A)$ :



$$\begin{aligned}
& 248832y^6 + y^4(1633536x^2 + 2141568x + 114192) \\
& + y^2(1684992x^4 + 2206464x^3 - 2691936x^2 - 5269430x \\
& - 1756248) \quad (562432x^6 + 64896x^5 - 1368432x^4 \\
& - 773312x^3 - 77064x^2 - 2808x - 35) = 0.
\end{aligned}$$

**Example 7.6.**

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 0 & \omega & 3 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$k = 27, \alpha = -1$ . The Cartesian equation of  $C(A)$ :

$$\begin{aligned}
& 54y^6 + y^4(414x^2 + 900x + 225) + y^2(606x^4 + 1500x^3 \\
& - 1710x^2 - 6906x - 4425) + (250x^6 + 600x^5 - 1395x^4 \\
& - 6122x^3 - 7800x^2 - 4320x - 896) = 0.
\end{aligned}$$

**Example 7.7.**

$$A = \begin{pmatrix} 3 & 3 & 3 \\ 0 & -2 & -31 \\ 0 & 0 & -1 \end{pmatrix}$$

$k = 27, \alpha = 1$ . The Cartesian equation of  $C(A)$ :

$$\begin{aligned}
& (2y - 3)^2 \left[ -80919x^4 + x^3(34992y - 52488) \right. \\
& + x^2(-225342y^2 - 61236y + 1097874) + x(34992y^3 \\
& - 215784y^2 - 559872y + 1207224) - (y + 3)(150823y^3 \\
& - 245025y^2 + 225261y - 124659) \left. \right] = 0.
\end{aligned}$$

**Example 7.8.**

$$A = \begin{pmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$k = 8$ ,  $\alpha = -2/3$ . Note that  $\alpha$  is at the cusp. The curve is a cardioid. The Cartesian equation of  $C(A)$ :

$$(x + 1)^2 [27(x^2 + y^2)^2 - 72(x^2 + y^2) - 64x - 16] = 0.$$

**Example 7.9.**

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 1 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$k = 1$ ,  $\alpha = 1 = \lambda_1$ .  $C(A)$  consists of an ellipse with foci at  $\omega$  and  $\omega^2$ , plus tangents from the external point  $\alpha = \lambda_1 = 1$ . The Cartesian equation of  $C(A)$ :

$$(x + \sqrt{2}y - 1)^2 (x - \sqrt{2}y - 1)^2 (4x^2 + y^2 + 4x) = 0.$$

**Example 7.10.**

$$A = \begin{pmatrix} 0 & 2r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$k = 4r^2$ ,  $\alpha = \lambda_1 = \lambda_2 = \lambda_3 = 0$ .  $C(A)$  consists of a circle of radius  $r$ , plus its center. The Cartesian equation of  $C(A)$ :

$$(x^2 + y^2)^2 (x^2 + y^2 - r^2) = 0$$

**Example 7.11.**

$$A \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

$k = 0$ ,  $\alpha$  is indeterminate. The Cartesian equation of  $C(A)$ :  
 $(2x + 1)^2 (x + \sqrt{3}y - 1)^2 (x - \sqrt{3}y - 1)^2 = 0$ .

**Example 7.12.** The class 3 curve for which  $\lambda_1 = 1$ ,  
 $\lambda_2 = 0$ ,  $\lambda_3 = -1$ ,  $\alpha = -2$ , and  $k = 1$ , and is not associated  
 with any matrix. The Cartesian equation is:

$$68y^6 + y^4(-564x - 840x^2 - 269) + y(-372x^4 - 48x^3 + 622x^2 + 448x + 88) + (4x^6 + 24x^5 + 43x^4 - 72x^2 - 64x - 16) = 0.$$

The pages that follow show graphs of the curves in  
 these 12 examples, numbered correspondingly.

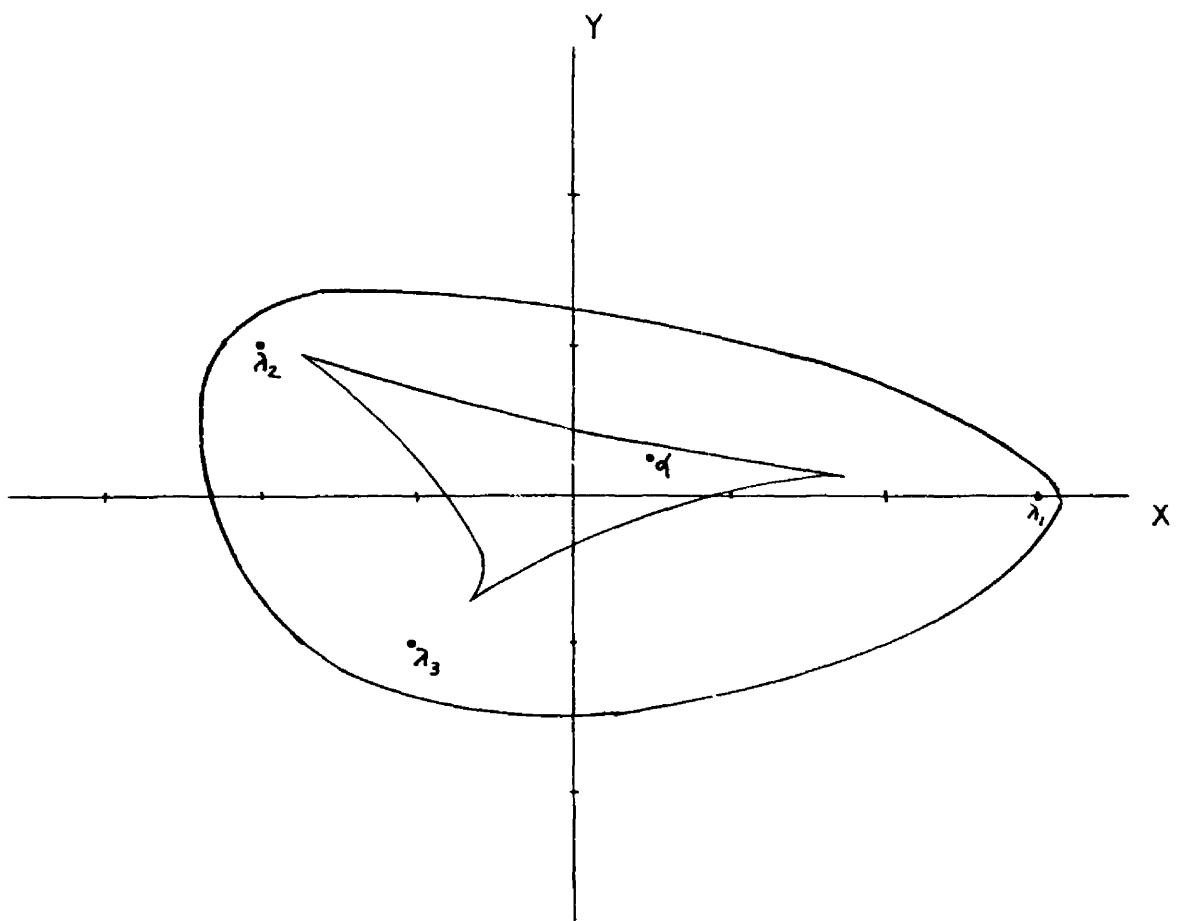


Figure 7.1

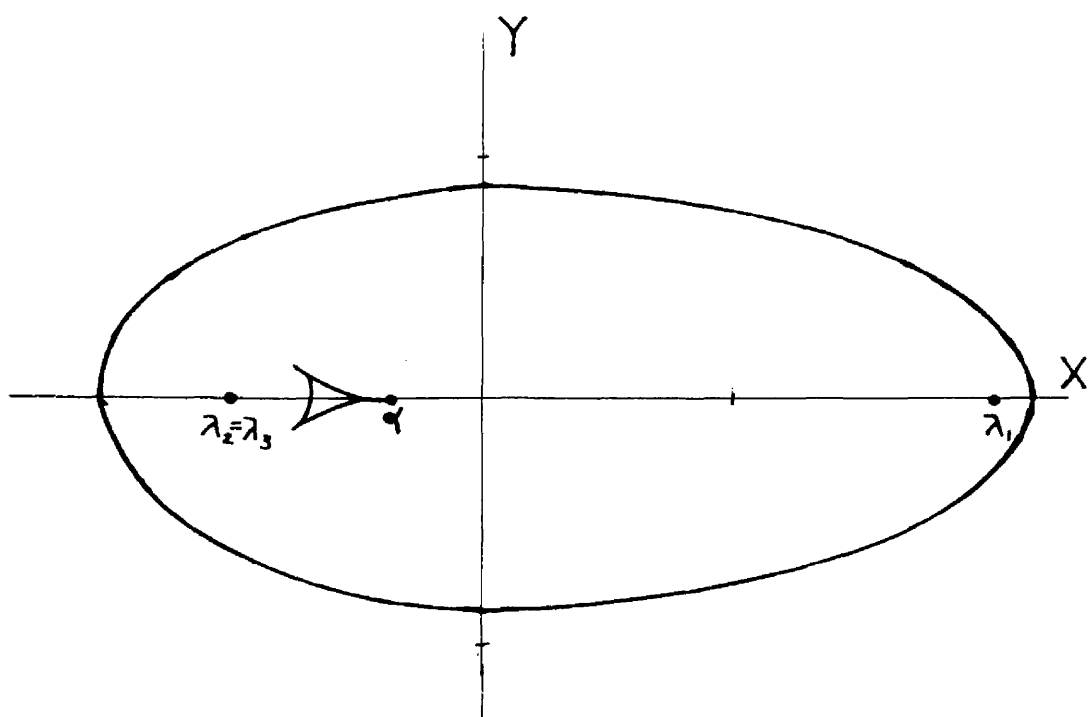


Figure 7.2

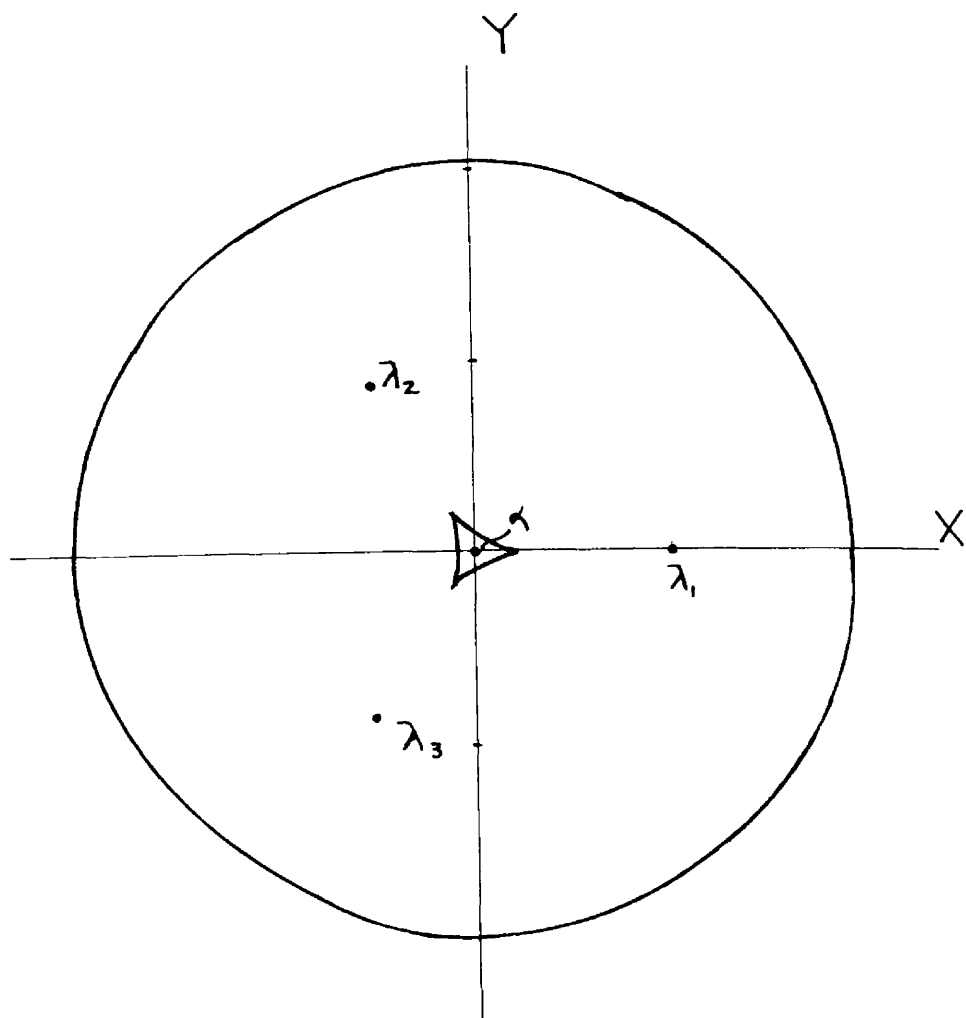


Figura 7.3

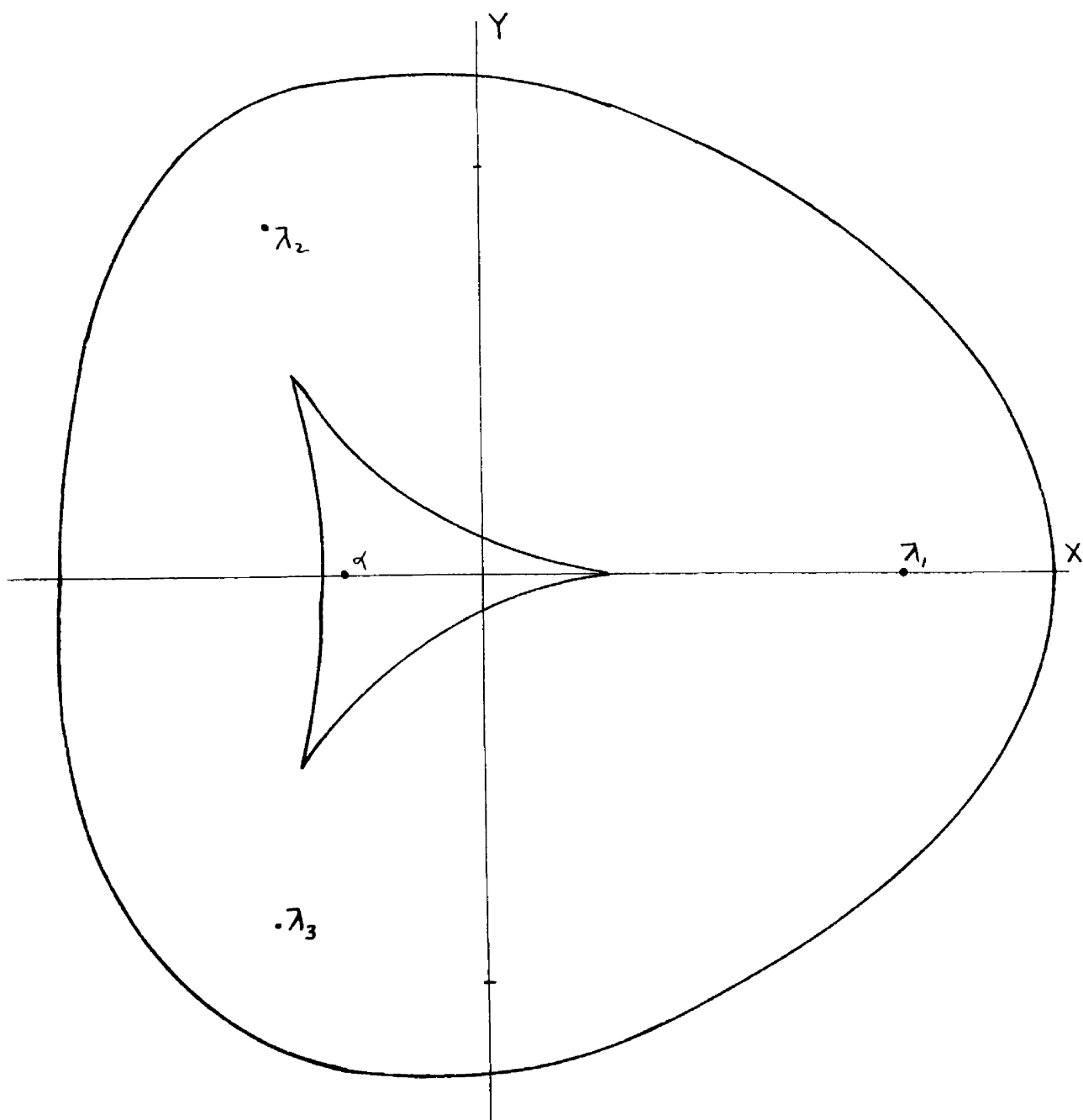


Figure 7.4

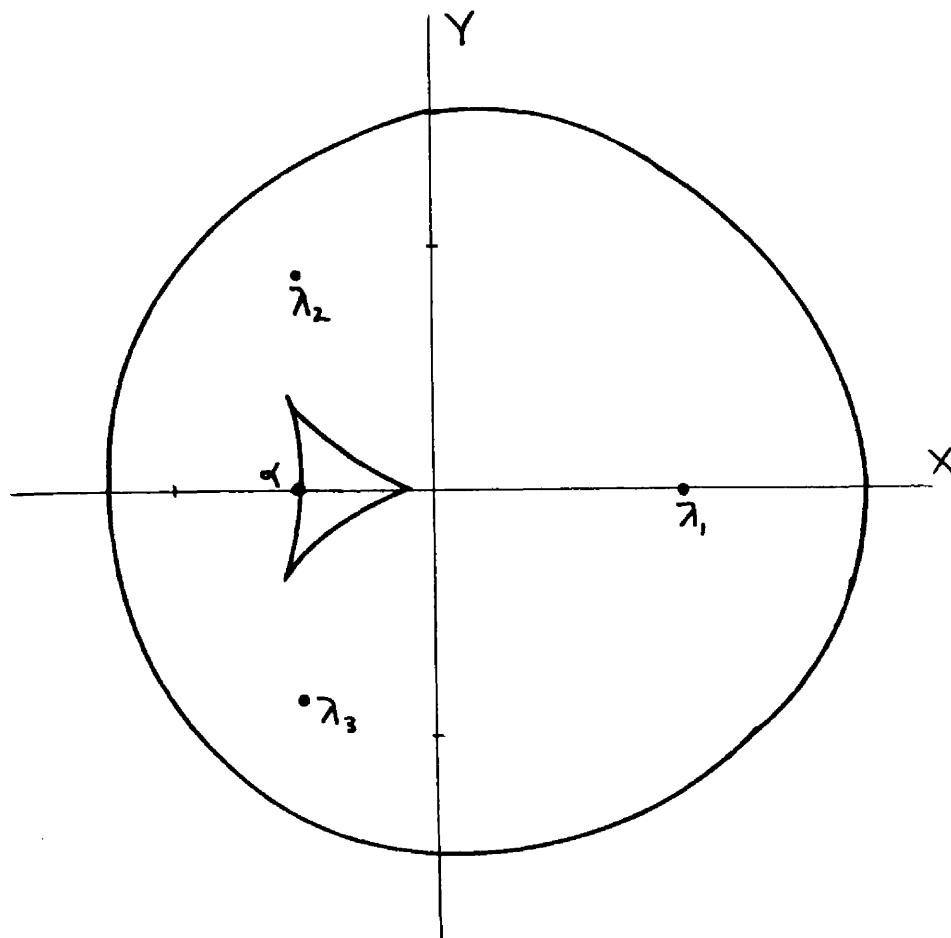


Figure 7.5



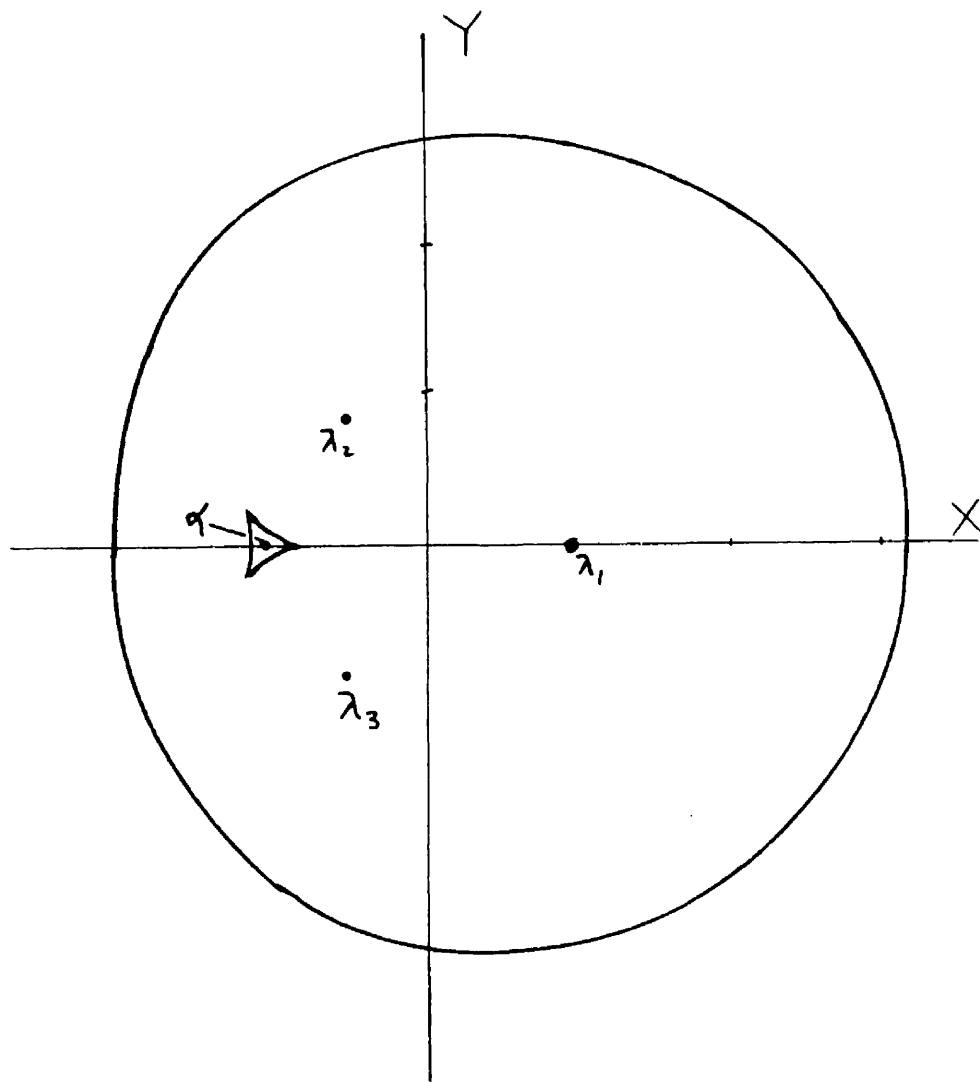


Figure 7.6

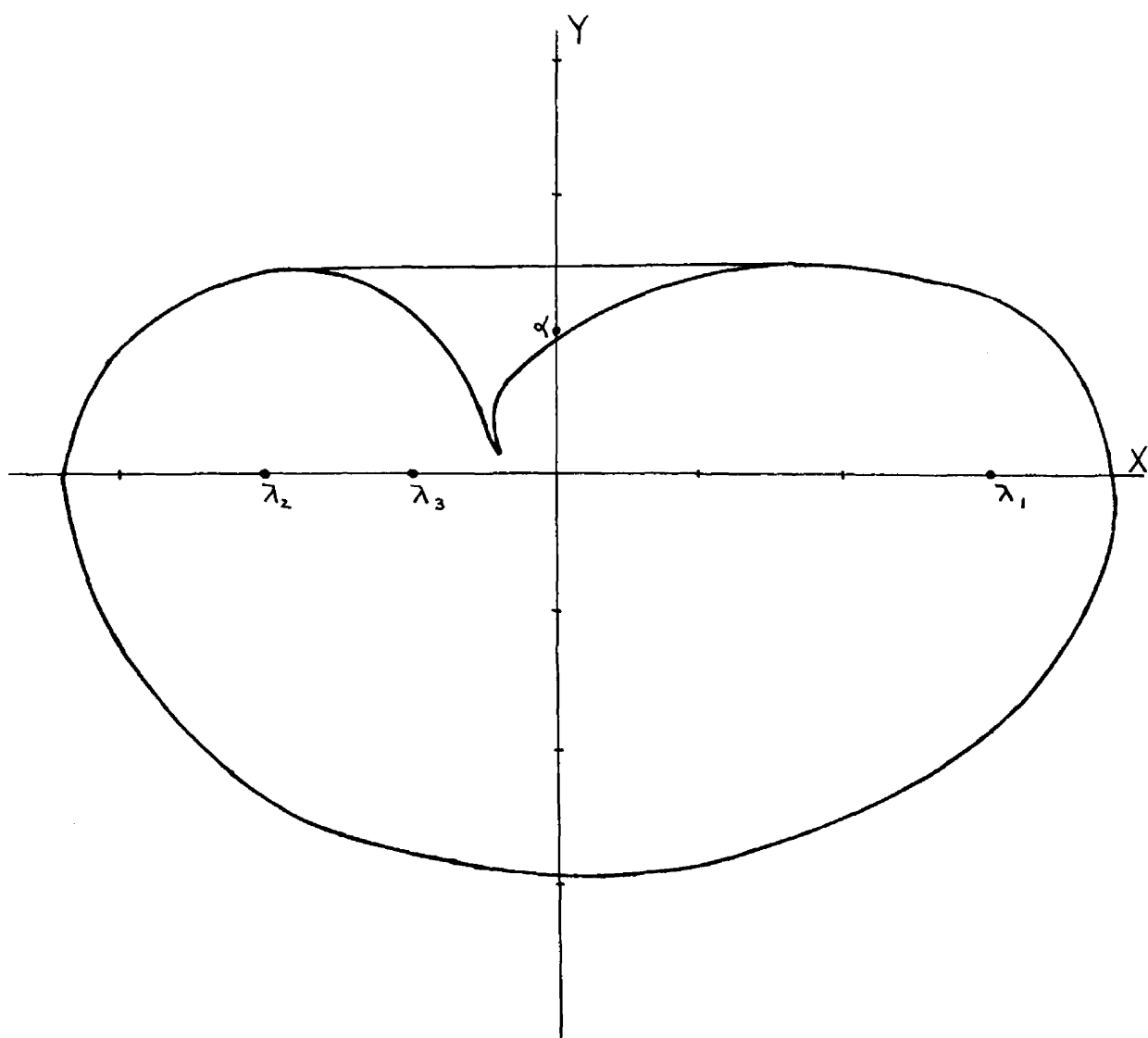


Figure 7.7

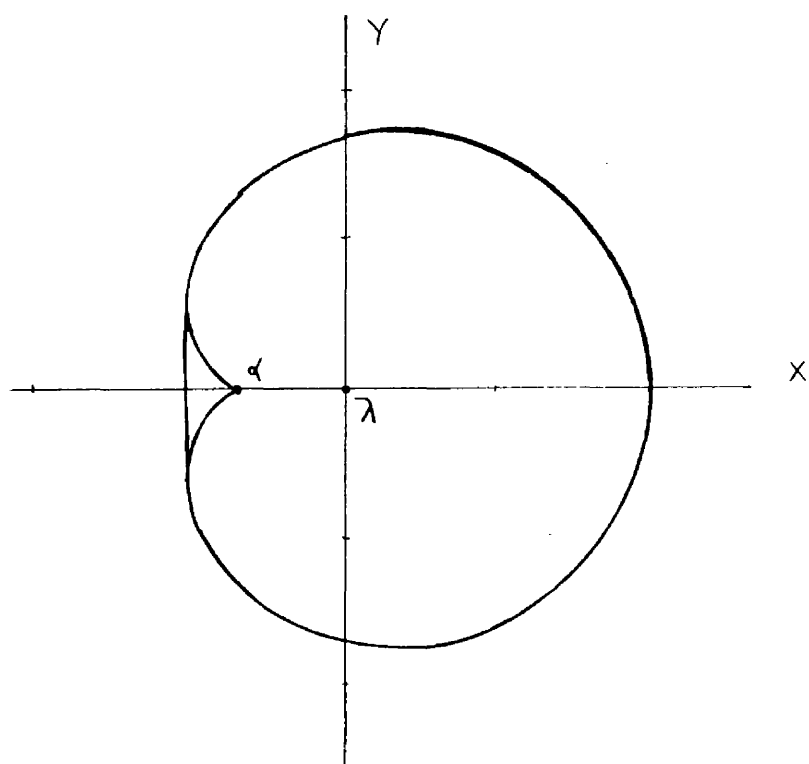


Figure 7.8

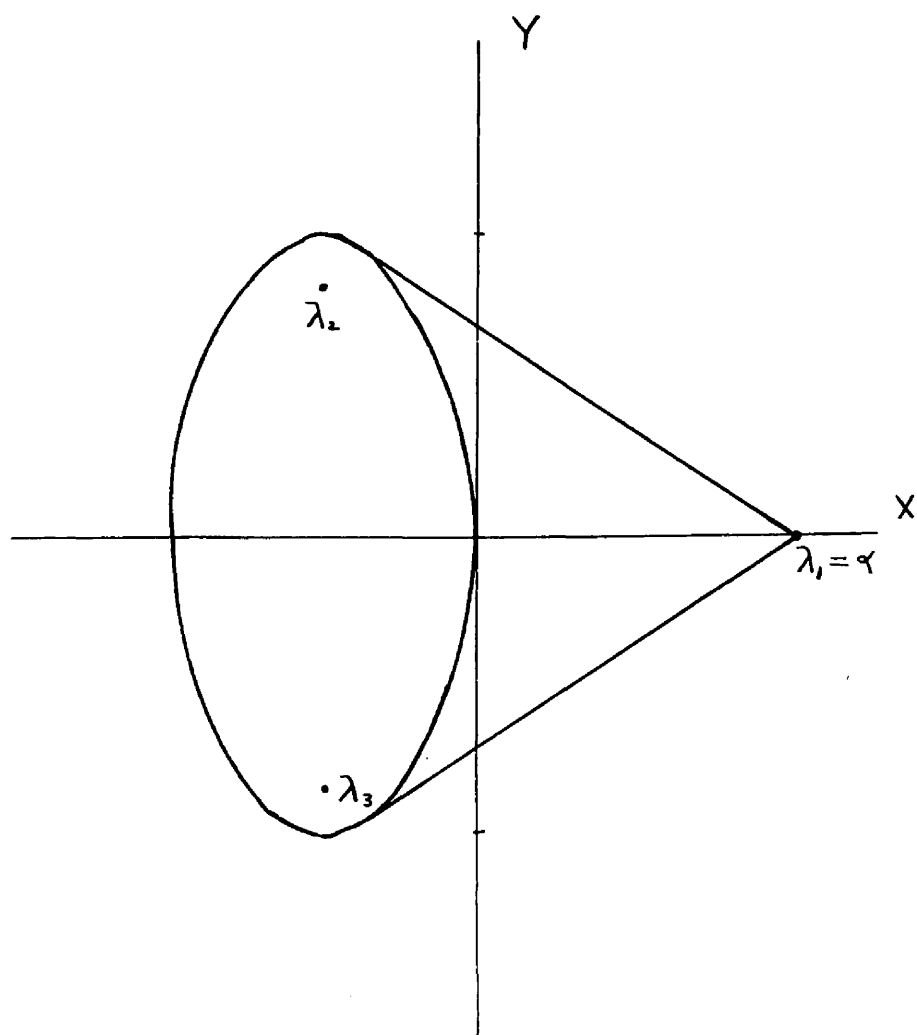
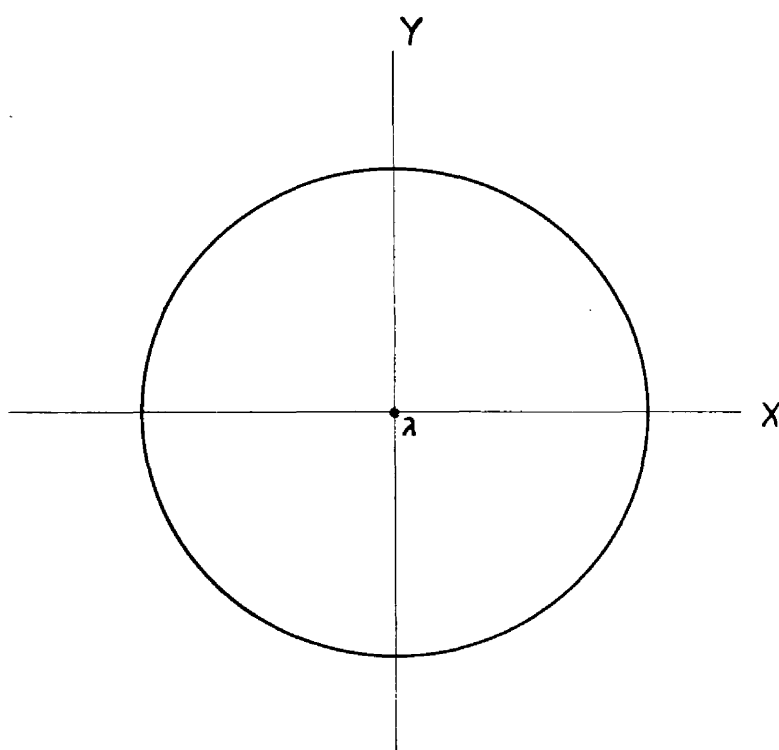
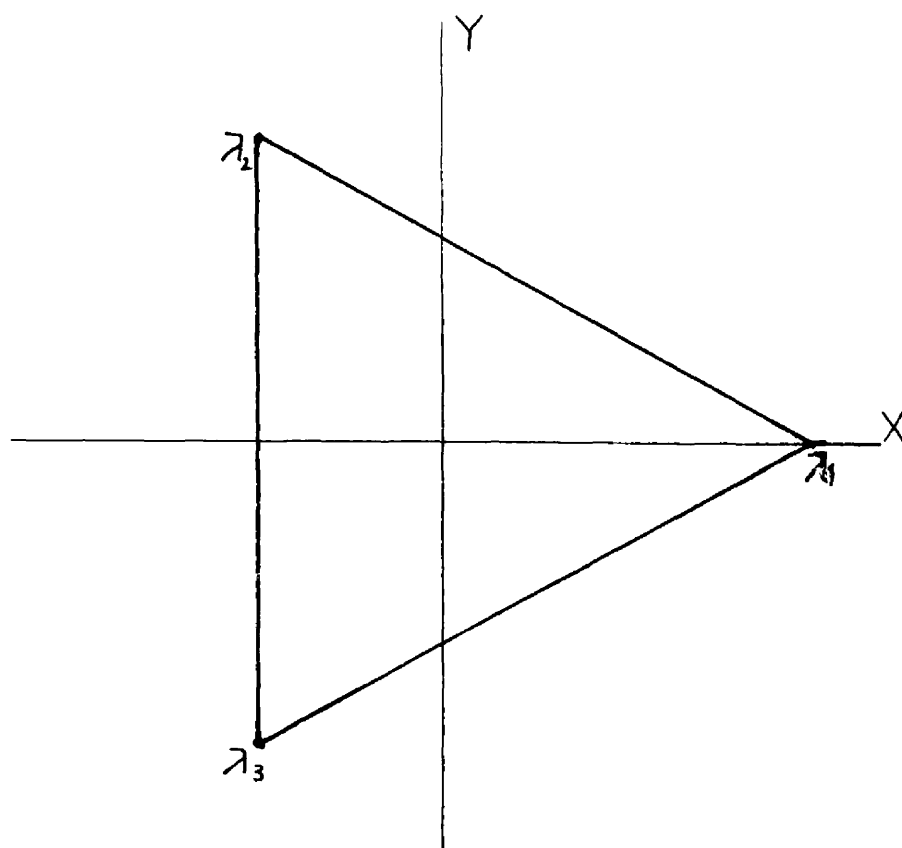


Figure 7.9



*Figure 7.10*



*Figure 7.11*

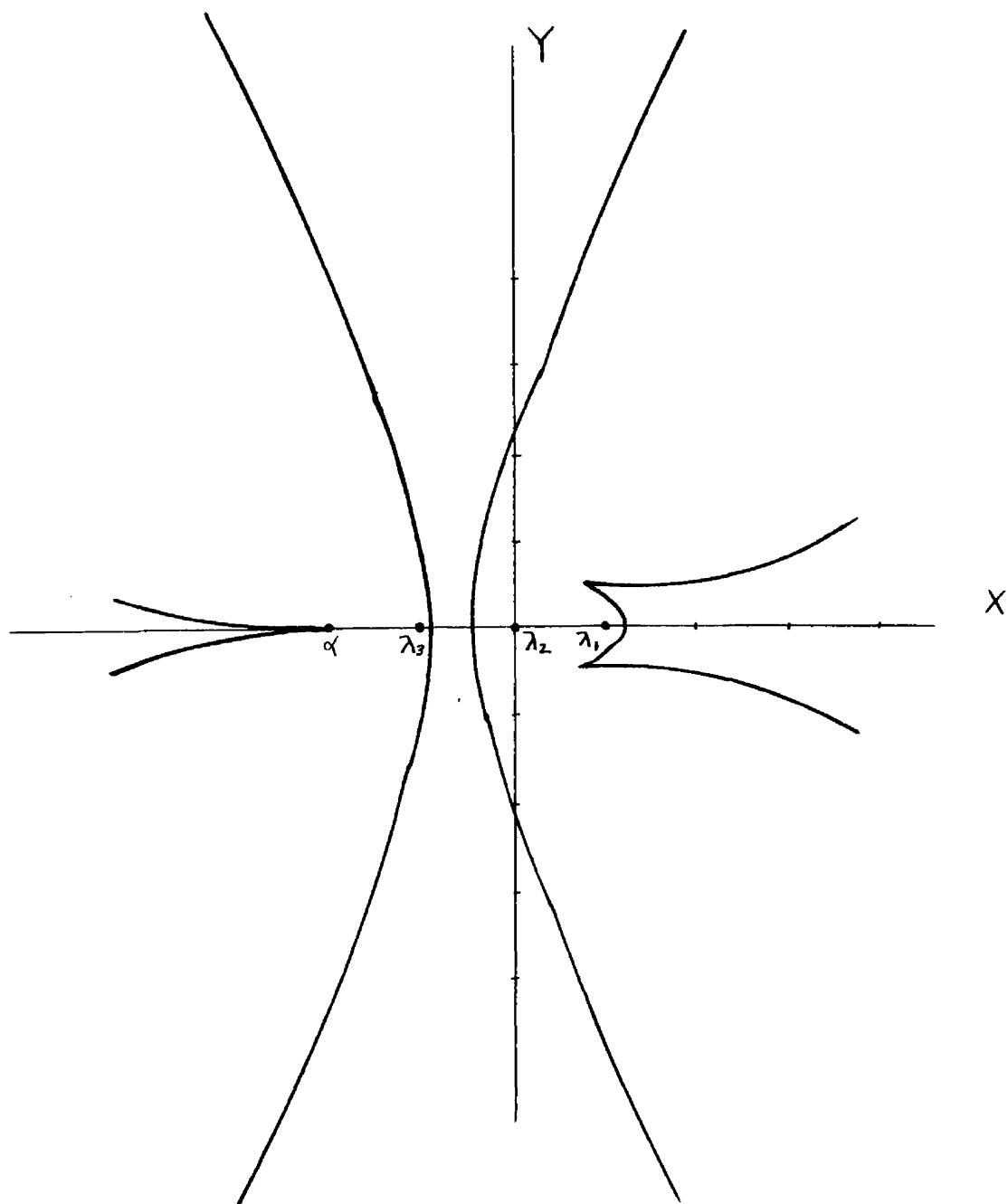


Figure 7.12

# VIII

## The Fourth Order Matrix

1. On the equation of C(A). Let A be the matrix

$$A = \begin{pmatrix} \lambda_1 & a_{12} & a_{13} & a_{14} \\ 0 & \lambda_2 & a_{23} & a_{24} \\ 0 & 0 & \lambda_3 & a_{34} \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}$$

Let the Hermitian decomposition of  $\bar{t}A$  be  $F_\theta + iG_\theta$ . Then

$$F_\theta = \begin{pmatrix} \frac{\lambda_1 \bar{t} + \bar{\lambda}_1 t}{2} & \frac{a_{12} \bar{t}}{2} & \frac{a_{13} \bar{t}}{2} & \frac{a_{14} \bar{t}}{2} \\ \frac{\bar{a}_{12} t}{2} & \frac{\lambda_2 \bar{t} + \bar{\lambda}_2 t}{2} & \frac{a_{23} \bar{t}}{2} & \frac{a_{24} \bar{t}}{2} \\ \frac{\bar{a}_{13} t}{2} & \frac{\bar{a}_{23} t}{2} & \frac{\lambda_3 \bar{t} + \bar{\lambda}_3 t}{2} & \frac{a_{34} \bar{t}}{2} \\ \frac{\bar{a}_{14} t}{2} & \frac{\bar{a}_{24} t}{2} & \frac{\bar{a}_{34} t}{2} & \frac{\lambda_4 \bar{t} + \bar{\lambda}_4 t}{2} \end{pmatrix}$$

The characteristic equation of  $F_\theta$  is :

$$\xi^4 + \frac{1}{t}(p_{20} \bar{t}^2 + p_{21} + p_{22} t^2) \xi^2 + \frac{1}{8}(p_{30} \bar{t}^3 + p_{31} \bar{t} + p_{32} t + p_{33} t^3) \xi + \frac{1}{16}(p_{40} \bar{t}^4 + p_{41} \bar{t}^2 + p_{42} + p_{43} t^2 + p_{44} t^4) = 0$$

where the  $p_{xi}$  are functions of the elements of A, defined below.

$$\text{Let } b_{ij} = -a_{ij} \bar{a}_{ij} ; i, j = 1, 2, 3, 4;$$



$$c_1 = -a_{23} \bar{a}_{24} a_{24}$$

$$c_3 = -a_{12} a_{14} a_{24}$$

$$c_2 = -a_{13} a_{14} a_{34}$$

$$c_4 = -a_{12} a_{13} a_{23}$$

$$\begin{aligned} d_1 = & a_{12} \bar{a}_{12} a_{34} \bar{a}_{34} + a_{13} \bar{a}_{13} a_{24} \bar{a}_{24} + a_{14} \bar{a}_{14} a_{23} \bar{a}_{23} \\ & - a_{12} \bar{a}_{13} a_{24} \bar{a}_{34} - \bar{a}_{12} a_{13} \bar{a}_{24} a_{34} \\ & - a_{13} \bar{a}_{14} \bar{a}_{23} a_{24} - \bar{a}_{13} a_{14} a_{23} \bar{a}_{24} \end{aligned}$$

$$d_2 = -a_{12} \bar{a}_{14} a_{23} a_{34}$$

$$\sigma_1 = -(\lambda_1 \bar{\lambda}_1 + \lambda_2 \bar{\lambda}_2 + \lambda_3 \bar{\lambda}_3 + \lambda_4 \bar{\lambda}_4)$$

$$\sigma_2 = (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4)$$

$$\sigma_3 = -(\lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4)$$

$$\sigma_4 = (\lambda_1 \lambda_2 \lambda_3 \lambda_4)$$

$$\sigma_5 = -(\lambda_1^2 \bar{\lambda}_1 + \lambda_2^2 \bar{\lambda}_2 + \lambda_3^2 \bar{\lambda}_3 + \lambda_4^2 \bar{\lambda}_4)$$

$$\sigma_6 = (\lambda_1 \lambda_2 \lambda_3 \bar{\lambda}_4 + \lambda_1 \lambda_2 \bar{\lambda}_3 \lambda_4 + \lambda_1 \bar{\lambda}_2 \lambda_3 \lambda_4 + \bar{\lambda}_1 \lambda_2 \lambda_3 \lambda_4)$$

$$\begin{aligned} \sigma_7 = & (\lambda_1 \lambda_2 \bar{\lambda}_3 \bar{\lambda}_4 + \lambda_1 \bar{\lambda}_2 \lambda_3 \bar{\lambda}_4 + \bar{\lambda}_1 \lambda_2 \lambda_3 \bar{\lambda}_4 \\ & + \lambda_1 \bar{\lambda}_2 \bar{\lambda}_3 \lambda_4 + \bar{\lambda}_1 \lambda_2 \bar{\lambda}_3 \lambda_4 + \bar{\lambda}_1 \bar{\lambda}_2 \lambda_3 \lambda_4) \end{aligned}$$

$$\beta_1 = \left( \sum b_{ij} \right)$$

$$\begin{aligned} \beta_2 = & -[b_{34}(\lambda_1 + \lambda_2) + b_{24}(\lambda_1 + \lambda_3) + b_{23}(\lambda_1 + \lambda_4) \\ & + b_{14}(\lambda_2 + \lambda_3) + b_{13}(\lambda_2 + \lambda_4) + b_{12}(\lambda_3 + \lambda_4)] \end{aligned}$$

$$\begin{aligned} \beta_3 = & b_{34} \lambda_1 \lambda_2 + b_{24} \lambda_1 \lambda_3 + b_{23} \lambda_1 \lambda_4 \\ & + b_{14} \lambda_2 \lambda_3 + b_{13} \lambda_2 \lambda_4 + b_{12} \lambda_3 \lambda_4 \end{aligned}$$

$$\begin{aligned} \beta_4 = & b_{34}(\lambda_1 \bar{\lambda}_2 + \bar{\lambda}_1 \lambda_2) + b_{24}(\lambda_1 \bar{\lambda}_3 + \bar{\lambda}_1 \lambda_3) \\ & + b_{23}(\lambda_1 \bar{\lambda}_4 + \bar{\lambda}_1 \lambda_4) + b_{14}(\lambda_2 \bar{\lambda}_3 + \bar{\lambda}_2 \lambda_3) \\ & + b_{13}(\lambda_2 \bar{\lambda}_4 + \bar{\lambda}_2 \lambda_4) + b_{12}(\lambda_3 \bar{\lambda}_4 + \bar{\lambda}_3 \lambda_4) \end{aligned}$$

$$\chi_1 = c_1 + c_2 + c_3 + c_4$$

$$\chi_2 = -(c_1 \lambda_1 + c_2 \lambda_2 + c_3 \lambda_3 + c_4 \lambda_4)$$

$$\gamma_3 = - (c_1 \bar{\lambda}_2 + \bar{c}_1 \lambda_1 + c_2 \bar{\lambda}_2 + \bar{c}_2 \lambda_2 + c_3 \bar{\lambda}_3 + \bar{c}_3 \lambda_3 + c_4 \bar{\lambda}_4 + \bar{c}_4 \lambda_4)$$

$$p_{20} = \sigma_2$$

$$p_{21} = \sigma_1 + \beta_1$$

$$p_{22} = \bar{p}_{20}$$

$$p_{30} = \sigma_3$$

$$p_{31} = \sigma_5 + \beta_2 + \gamma_1$$

$$p_{32} = \bar{p}_{31}$$

$$p_{33} = \bar{p}_{30}$$

$$p_{40} = \sigma_4$$

$$p_{41} = \sigma_6 + \beta_3 + \gamma_2 + d_2$$

$$p_{42} = \sigma_7 + \beta_4 + \gamma_3 + d_1$$

$$p_{43} = \bar{p}_{41}$$

$$p_{44} = \bar{p}_{40}$$

Now follow the usual procedure: replace  $\xi$  by  $\frac{1}{2}(z\bar{t} + \bar{z}t)$ , multiply by  $(2\bar{t})^4$ , substitute  $s = \bar{t}^2$ , and collect terms in  $s$ , to obtain

$$u_0 s^4 + u_1 s^3 + u_2 s^2 + \bar{u}_1 s + \bar{u}_0 = 0$$

where  $u_0 = z^4 + p_{20} z^2 + p_{30} z + p_{40}$

$$u_1 = 4z^3 \bar{z} + p_{21} z^2 + 2p_{20} z \bar{z} + p_{31} z + p_{30} \bar{z} + p_{41}$$

$$u_2 = 6z^2 \bar{z}^2 + \bar{p}_{20} z^2 + 2p_{21} z \bar{z} + p_{20} \bar{z}^2 + \bar{p}_{31} z + p_{31} \bar{z} + p_{42}$$

The desired equation of  $C(A)$  is then obtained by equating to zero the discriminant of this polynomial in  $s$ .<sup>1</sup> The

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<sup>1</sup> Burnside and Panton, op. cit., pp. 121, 144.

resulting equation in terms of the  $u$  is

$$4(u_0 \bar{u}_0 - 3u_1 \bar{u}_1 + u_2^2)^3 - (72u_0 \bar{u}_0 u_2 + 9u_1 \bar{u}_1 u_2 - 27u_0 \bar{u}_1^2 - 27\bar{u}_0 u_1^2 - 2u_2^3)^2 = 0$$

The  $u_i$  may be replaced by their values in terms of  $z$ , and  $\bar{z}$  by  $x + iy$  to obtain the Cartesian equation. This calculation has not been performed beyond a point sufficient to verify that it is of the 12th degree, a fact already known from the Plücker formula. The coefficient of  $\bar{z}^{12}$  is the discriminant of the characteristic equation of  $A$ , so that, as in the third order case, if  $A$  has a multiple root,  $C(A)$  passes through the circular points at infinity.

2. Plücker's formulas and Klein's equation. The following table lists the types of curves of class 4 obtained, according to their Plucker numbers.

	$n$	$m$	$\delta$	$\kappa$	$\tau$	$l$	$k + 2d$
$\{a\}$	12	4	28	24	0	0	8
$\{b\}$	10	4	16	18	1	0	6
$\{c\}$	8	4	8	12	2	0	4
$\{d\}$	6	4	4	6	3	0	2

As in the third order case, the table does not exhaust the possibilities, since it does not include the degenerate class four curves for which the Plücker numbers have no significance. These will be described separately.

(a) Klein's equation,  $k + 2d = 8$ , has the solutions in non-negative integers  $(k,d) = (8,0), (6,1), (4,2), (2,3), (0,4)$ . Example 1 illustrates the  $(8,0)$  case. Example 2 illustrates the  $(4,2)$  case; in order to see this it is necessary to note that the points  $(\pm \frac{1}{2}, 0)$  are triple points and must therefore be counted as 2 cusps and 1 node,<sup>2</sup> and that the points  $(\pm \sqrt{2}/4, 0)$  are real acnodes.

(b) Here,  $k + 2d = 6$ , with solutions  $(6,0), (4,1), (2,2), (0,3)$ . In connection with this and the other cases with bitangents, it is necessary to distinguish between two types of bitangents, according as the corresponding double root of  $F_\theta$  is of the type  $\xi_{\theta 1} = \xi_{\theta 2} > \xi_{\theta 3} > \xi_{\theta 4}$  or of the type  $\xi_{\theta 1} > \xi_{\theta 2} = \xi_{\theta 3} > \xi_{\theta 4}$ . Call the bitangent in the first case an external bitangent, and in the other, an internal bitangent. It is not easy to determine which of the possibilities with reference to the external or internal character of the bitangents can actually occur. In the case of a single bitangent, Example 3 illustrates the solution  $(k,d) = (6,0)$  with an internal bitangent; apparently there is no curve with a single external bitangent.

(c) With two bitangents there are three possibilities: that both are external; that one is external and one is internal; that both are internal. Apparently only

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<sup>2</sup> Hilton, op. cit., p. 116.

the first possibility actually exists. The Klein equation,  $k + 2d = 4$ , has the solutions  $(4,0)$ ,  $(2,1)$ ,  $(0,2)$ . Example 4 illustrates the case  $(0,2)$  with both bitangents external. The two acnodes are  $(0, \pm (\sqrt{5} - 1)/4)$ .

(d) Here  $k + 2d = 2$ , which has solutions  $(2,0)$  and  $(0,1)$ . Example 5 illustrates the  $(2,0)$  case, and has two external bitangents and one internal bitangent. Example 6 illustrates the same  $(2,0)$  case, but with three external bitangents (coincident).

This exhausts the non-degenerate types.

(e) Let one line of the family of four lines pass through a fixed point for all  $\theta$ . Then evidently the remaining three lines envelope a class 3 curve, and  $C(A)$  consists of this class three curve plus the tangents to it from the fixed point. As these tangents are also tangent to  $C(A)$  at the fixed point, they are bitangents to  $C(A)$ , and are three in number, since they are the tangents from a point to a class 3 curve. Finally, the tangents are real or imaginary according as the fixed point is (1) outside or on the convex boundary of the class 3 curve; (2) inside the convex boundary. And since there are essentially two types of non-degenerate class 3 curves, there are four different situations that arise. Example 7 shows one of these. Note that it is sufficient but not necessary, in order that  $C(A)$  be of type (e), for the matrix  $A$  to be of the form

$$A = \begin{pmatrix} T & 0 \\ 0 & a \end{pmatrix}$$

where  $T$  is a non-normal third order matrix,  $C(T)$  is a non-degenerate class 3 curve, and  $a$  is the fixed point.

(f) Let  $C'(a)$  be factorable into two distinct curves. Then, since the total class is 4, and the class of each factor must be at least 2, it is clear that each factor is of class 2, and therefore an ellipse. Now two ellipses have four common tangents, which are bitangents to  $C(A)$ . These bitangents are all real, or only two real, or none real, according as the two ellipses are non-overlapping, overlapping, or such that one is interior to the other. Example 8 illustrates the first of these.

In order to obtain a curve  $C(A)$  of this type, it is sufficient but not necessary that  $A$  be of the form:

$$A = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

where  $T_1$  and  $T_2$  are non-normal second order matrices.

(g) Let two of the four lines of the family which envelopes  $C(A)$  pass through fixed points  $a_1$  and  $a_2$  for all  $\theta$ . Then the remaining two lines envelope a class 2 curve; that is, an ellipse. Then  $C(A)$  consists of this ellipse, plus the tangents to it from  $a_1$ , plus the tangents to it

from  $a_2$ , plus the line joining  $a_1$  and  $a_2$ . These five lines are bitangents to  $C(A)$ . The different cases arise from the varying positions which  $a_1$  and  $a_2$  may assume with respect to the ellipse. Example 9 illustrates the case in which both points are outside the ellipse.

As before, it is sufficient but not necessary that  $A$  be of the form

$$A = \begin{pmatrix} T & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix}$$

where  $T$  is a non-normal second order matrix, and  $a_1$  and  $a_2$  are the fixed points.

(h) For the discussion of the normal case, see section IV.

3. Murnaghan's equation. The Murnaghan p-equation is given in terms of notation listed above:

$$\begin{aligned} & p_1 p_2 p_3 p_4 + \frac{1}{4} (b_{34} p_1 p_2 + b_{24} p_1 p_3 + b_{23} p_1 p_4 \\ & + b_{14} p_2 p_3 + b_{13} p_2 p_4 + b_{12} p_3 p_4) + \frac{1}{8} [p_1 (c_1 \bar{t} + \bar{c}_1 t) \\ & + p_2 (c_2 \bar{t} + \bar{c}_2 t) + p_3 (c_3 \bar{t} + \bar{c}_3 t) + p_4 (c_4 \bar{t} + \bar{c}_4 t)] \\ & + \frac{1}{16} [d_1 + d_2 \bar{t}^2 + \bar{d}_2 t^2] = 0 \end{aligned}$$

If the term  $p_1 p_2 p_3 p_4$  is deleted, and the remaining terms equated to zero, the result is a class two curve which is the fourth order analogue of the point  $\alpha$  in the third order case. The analogy lies in the fact that a tangent to  $C(A)$  through a focus,  $\lambda_1$ , is also tangent to this class 2 curve.

The argument is the same: for a tangent through a focus,  $p_i = 0$ , so that the term  $p_1 p_2 p_3 p_4 = 0$ , an equation which implies the vanishing of the remaining terms. The situation illustrates an observation of Hilton's that the tangents to a class  $m$  curve from its foci are also tangent to a certain class  $(m - 2)$  curve.<sup>3</sup>

It has been shown that a class 2 curve can be expressed  $p_\alpha p_\beta - k = 0$ , where  $\alpha$  and  $\beta$  are the foci and  $k$  is a real number. It will now be established that the equation above can be written in the form

$$p_1 p_2 p_3 p_4 - \frac{1}{4} k_1 (p_\alpha p_\beta - k_2) = 0$$

for points  $\alpha$  and  $\beta$ , and real numbers  $k_1$  and  $k_2$  properly chosen. Since  $p_i = \xi - \frac{1}{2}(\lambda_i \bar{t} + \bar{\lambda}_i t)$ , and  $\xi = \frac{1}{2}(z\bar{t} + \bar{z}t)$ , we may write

$$p_i = \frac{1}{2} [(z - \lambda_i)\bar{t} + (\bar{z} - \bar{\lambda}_i)t]$$

Now leaving the term  $p_1 p_2 p_3 p_4$  undisturbed, substitute these values into the general equation above, and collect terms in  $t$ :

$$p_1 p_2 p_3 p_4 + \frac{1}{16} \left\{ \begin{aligned} & [\beta_1 z^2 + (\beta_2 + \gamma_1)z \\ & + (\beta_3 + \gamma_2 + d)] \bar{t}^2 + 2\beta_1 z\bar{z} + (\bar{\beta}_2 + \bar{\gamma}_1)z \\ & + (\beta_2 + \gamma_1)\bar{z} + (\beta_4 + \gamma_3 + d) \\ & + [\bar{\beta}_1 \bar{z}^2 + (\bar{\beta}_2 + \bar{\gamma}_1)\bar{z} + (\bar{\beta}_3 + \bar{\gamma}_2 + d_1)] t^2 \end{aligned} \right\} = 0.$$

Similarly, substitute  $p_\alpha = \frac{1}{2}[(z - \alpha)\bar{t} + (\bar{z} - \bar{\alpha})t]$

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<sup>3</sup> Hilton and Jervis, op. cit., p. 430.



and  $p_\theta = \frac{1}{2}[(z - \beta)\bar{t} + (\bar{z} - \bar{\beta})t]$  in the proposed simpler form and collect terms in  $t$ :

$$p_1 p_2 p_3 p_4 + \frac{1}{i\epsilon} (-k_1) \left\{ [z^2 - (\alpha + \beta)z + \alpha\beta] \bar{t}^2 + 2z\bar{z} - (\bar{\alpha} + \bar{\beta})z - (\alpha + \beta)\bar{z} + \alpha\bar{\beta} + \bar{\alpha}\beta + [\bar{z}^2 - (\bar{\alpha} + \bar{\beta})\bar{z} + \bar{\alpha}\bar{\beta}] t^2 - 4k_2 \right\} = 0.$$

On comparing coefficients, it may be seen that this equation is identical with the preceding one if

$$k_1 = -\beta_1$$

$$k_1(\alpha + \beta) = (\beta_2 + \gamma_1)$$

$$-k_1 \alpha\beta = \beta_3 + \gamma_2 + d_2$$

$$-k_1(\alpha\bar{\beta} + \bar{\alpha}\beta) + 4k_1 k_2 = \beta_4 + \gamma_3 + d_1$$

Reference to the definitions reveals that  $\beta_1, \beta_4, \gamma_3$ , and  $d_1$  are real, and that  $k_1 \geq 0$ . The number  $k_1$  being known from the first equation, the second and third may always be solved for  $\alpha$  and  $\beta$ , and then the fourth equation gives  $k_2$ .

### 3. Numerical examples. Example 8.1.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The Cartesian equation of  $C(A)$  is  $4I^3 - J^2 = 0$ , where

$$\begin{aligned} I &= 121x^4 - 48x^3y + 346x^2y^2 - 48xy^3 + 121y^4 \\ &\quad - 14x^3 + 14x^2y - 14xy^2 + 14y^3 + 38xy \\ &\quad + 34x - 34y + 58 \end{aligned}$$

$$\begin{aligned}
J = & 2446x^6 - 900x^5y + 15906x^4y^2 - 1800x^3y^3 \\
& + 15906x^2y^4 - 900xy^5 + 2446y^6 - 580x^5 \\
& + 652x^4y + 1382x^3y^2 - 1382x^2y^3 - 652xy^4 \\
& + 580y^5 - 5745x^4 + 1752x^3y - 25872x^2y^2 \\
& + 1752xy^3 - 5745y^4 + 1423x^3 - 3171x^2y \\
& + 3171xy^2 - 1423y^3 + 186xy + 2904x^2 \\
& + 2904y^2 - 444x + 444y - 173
\end{aligned}$$

**Example 8.2.**

$$A = \begin{pmatrix} 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Example 8.3.**

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

**Example 8.4.**

$$A = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Example 8.5.**

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Example 8.6.**

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

**Example 8.7.**

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & & 1 & 0 \\ 0 & 0 & & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

**Example 8.8.**

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Note that this matrix is very nearly like the matrix of Example 1, and that the curves are "near".

Example 8.9.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

The pages that follow show graphs of the curves of these examples, numbered to correspond.

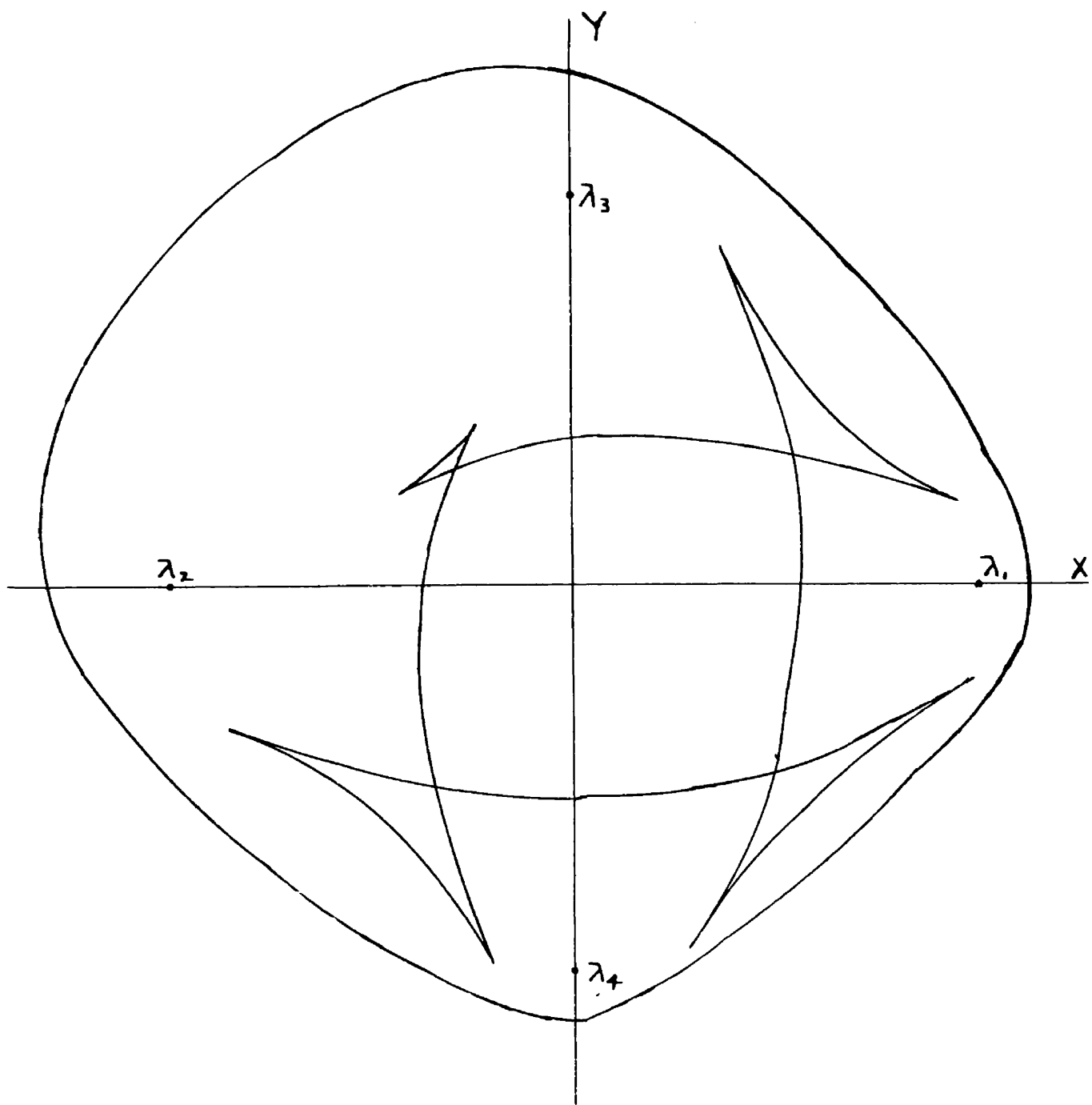


Figure 8.1

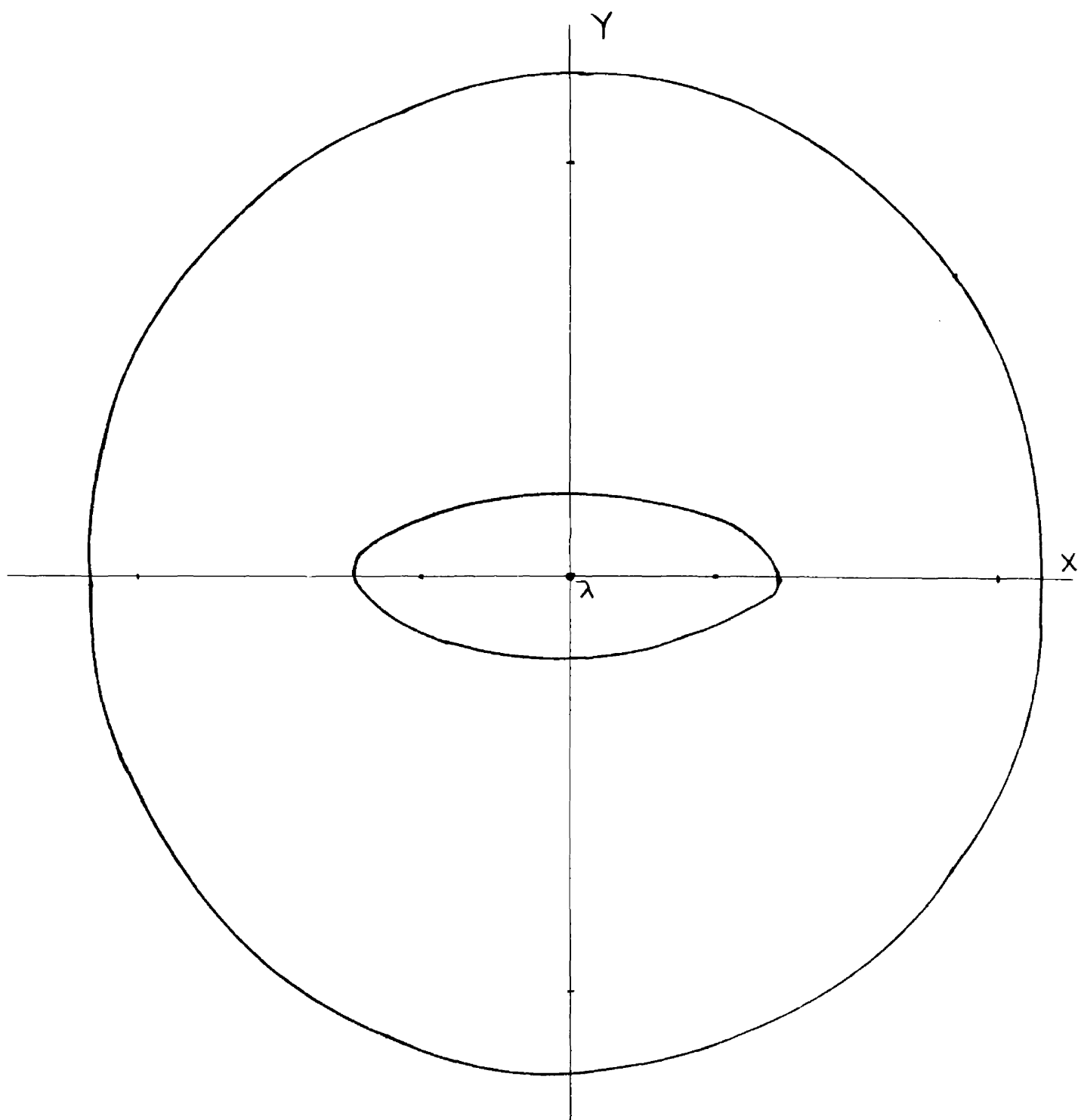


Figure 8.2

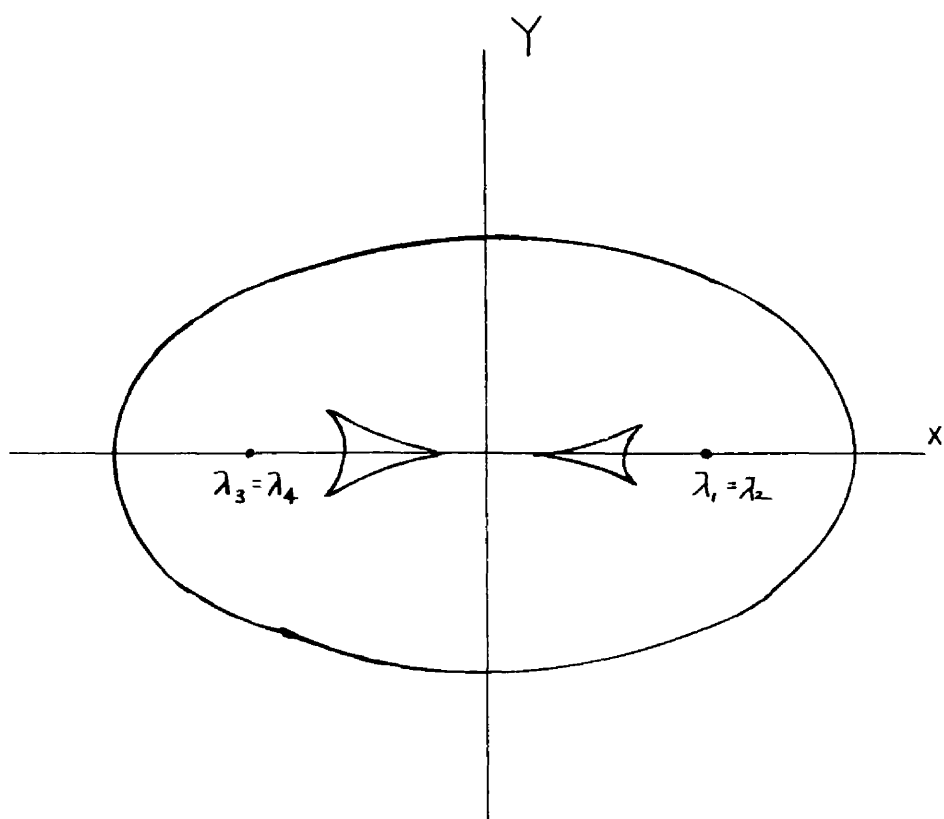


Figure 8.3

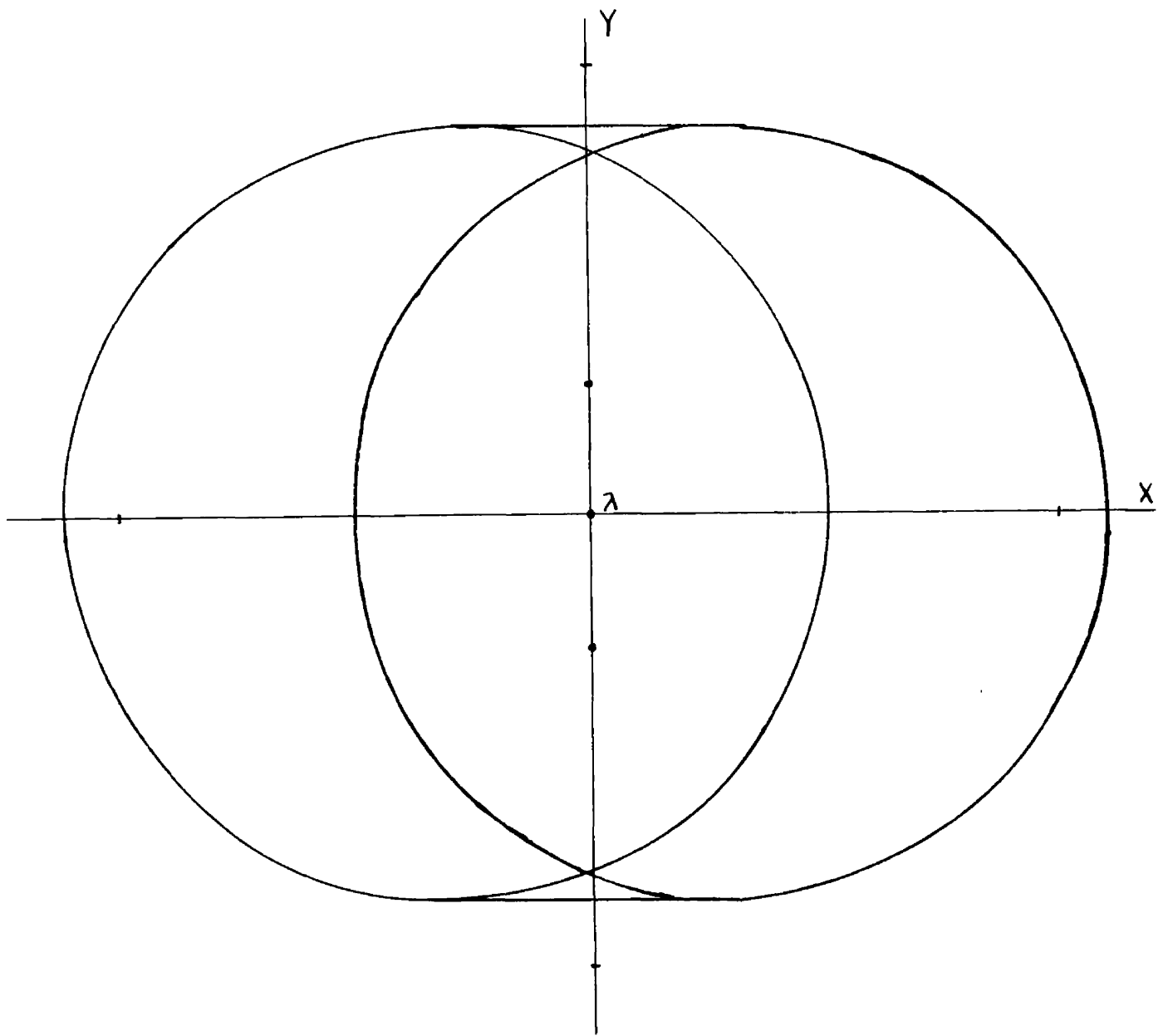


Figure 8.4



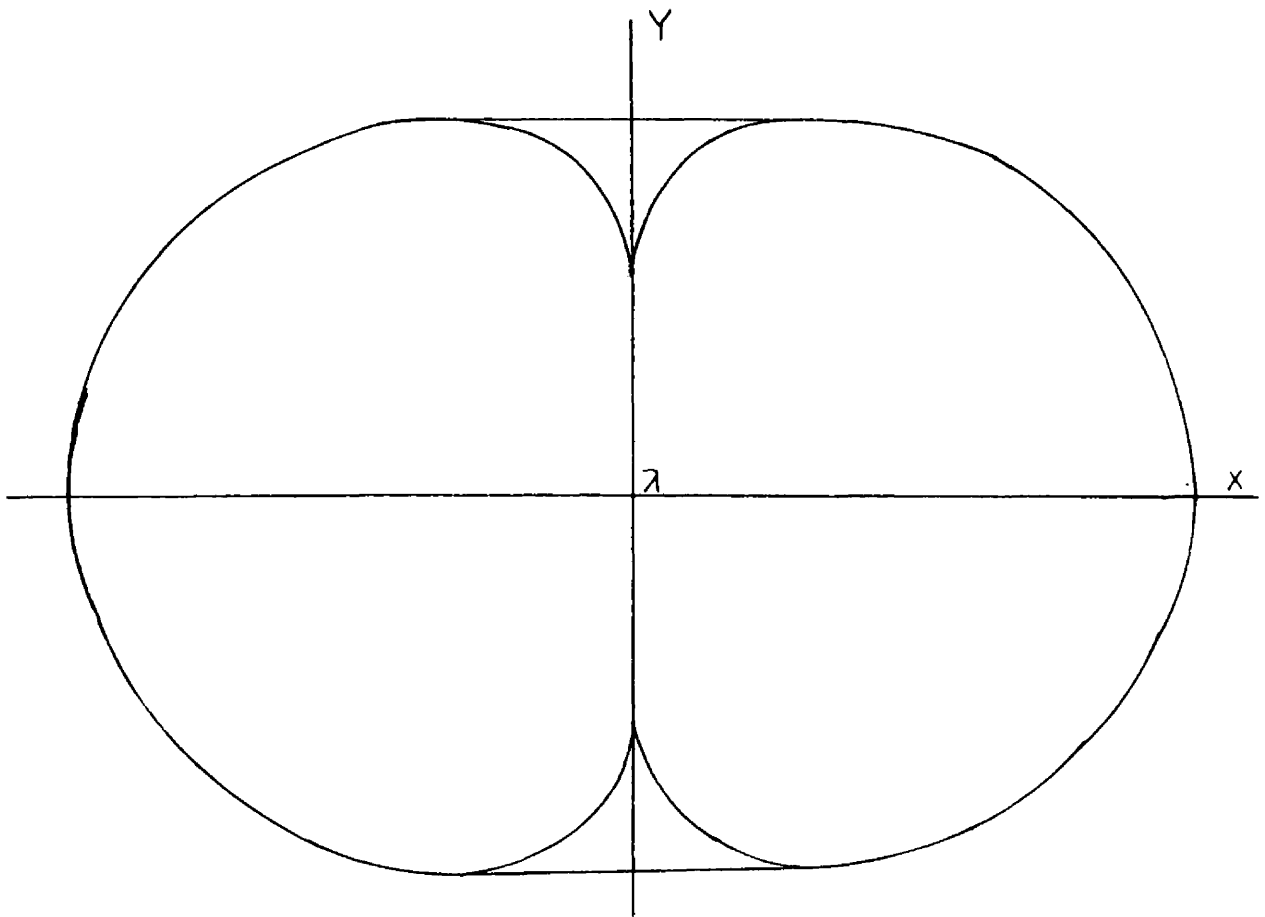


Figure 8.5

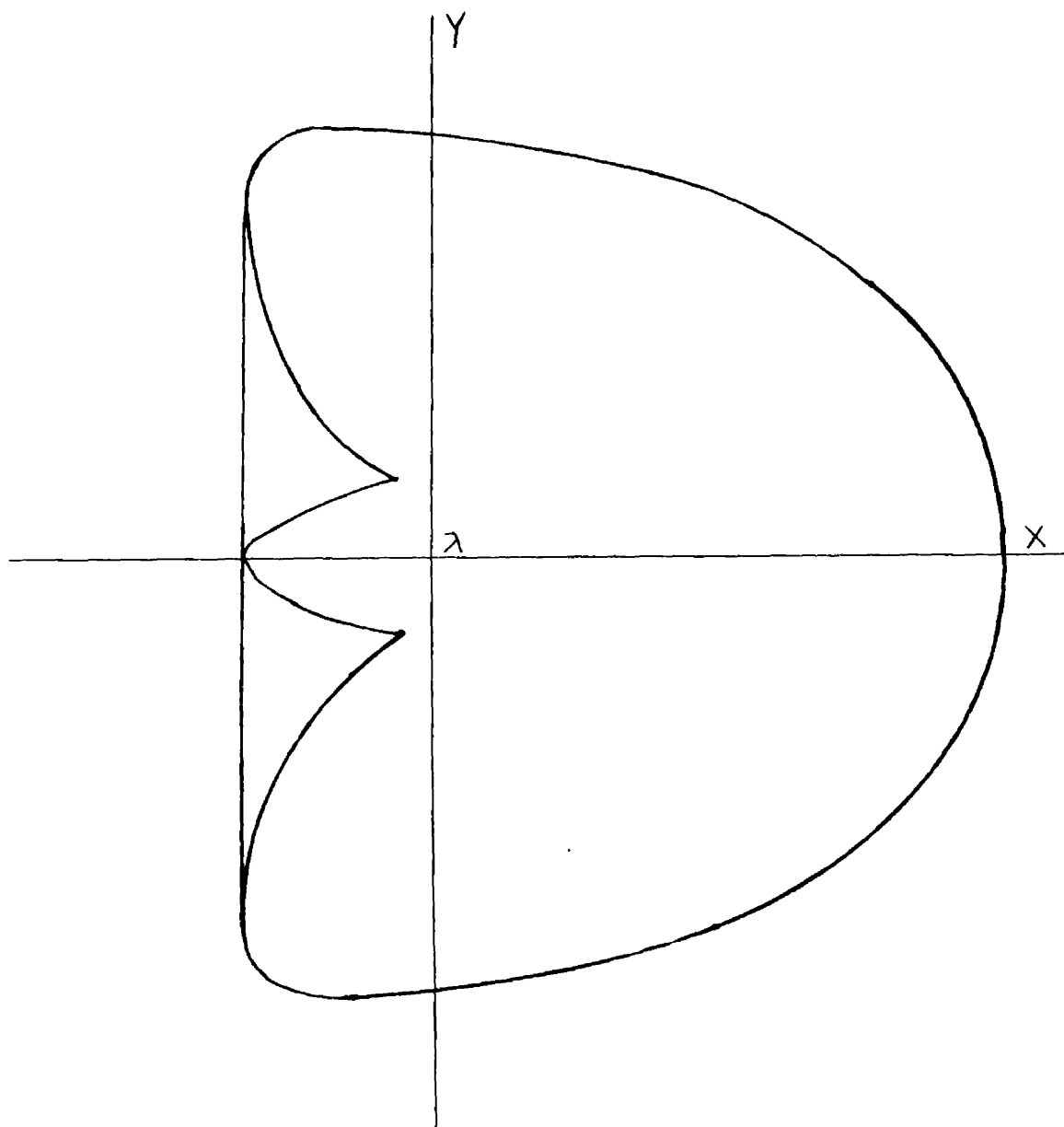


Figure 8.6

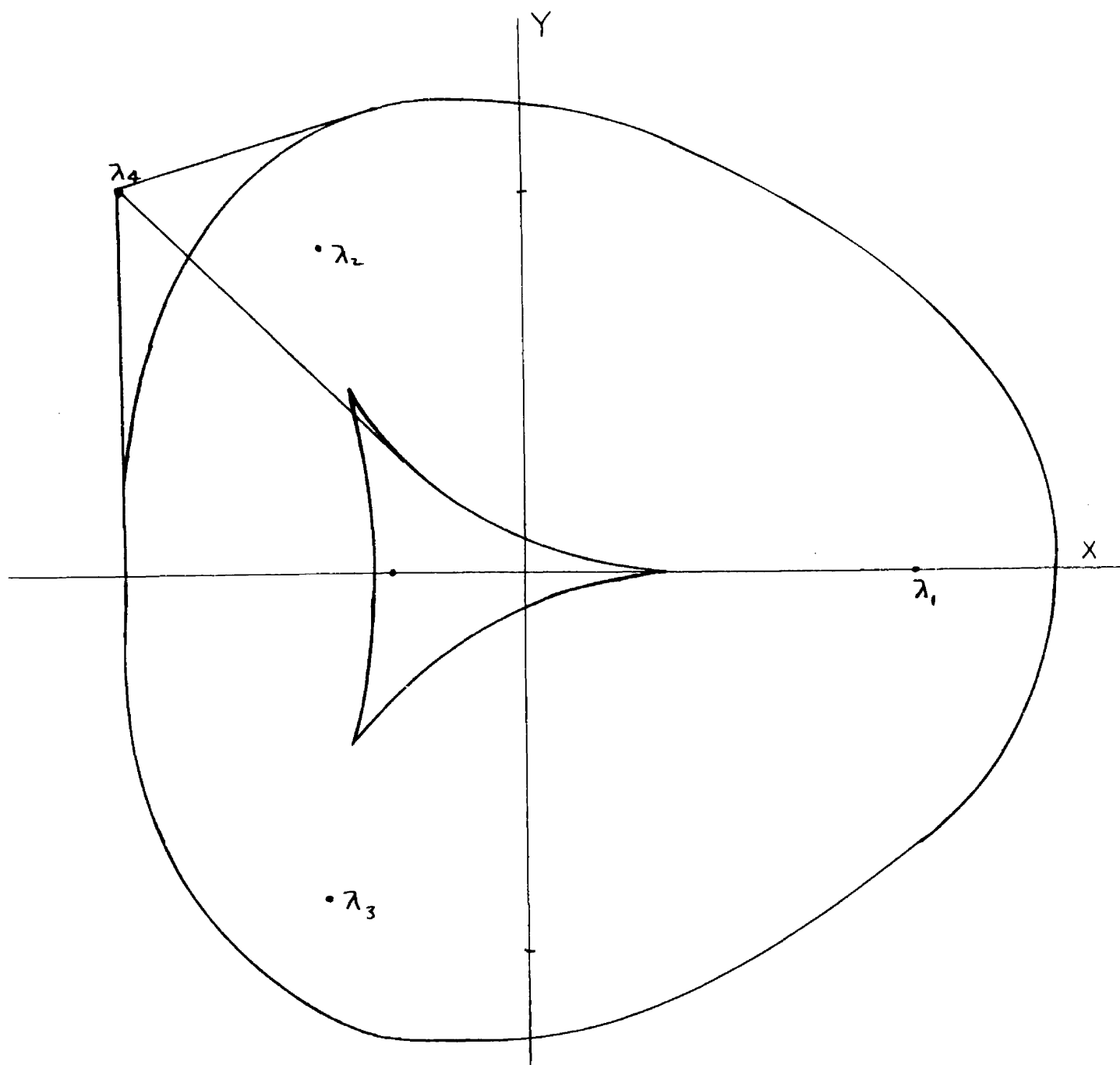
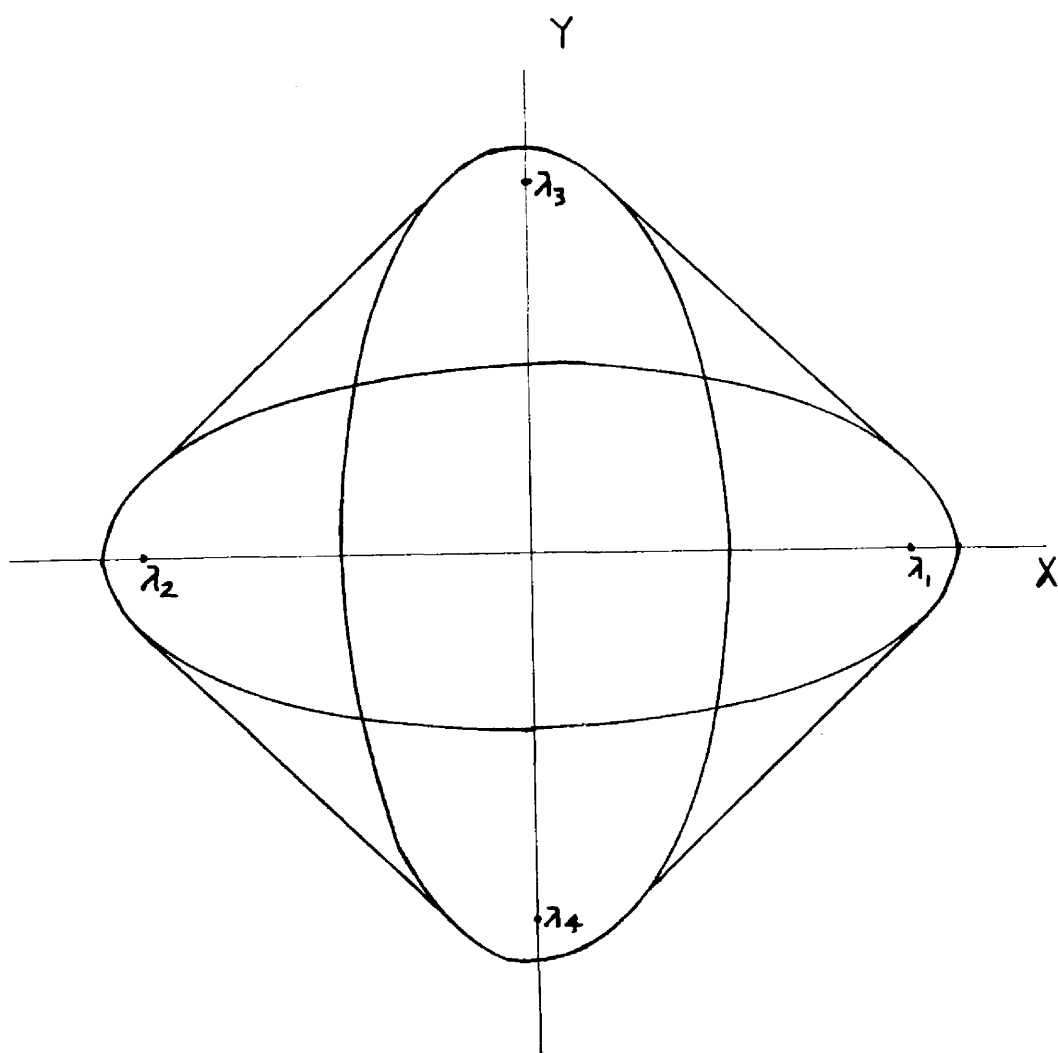


Figure 8.7



*Figure 8.8*

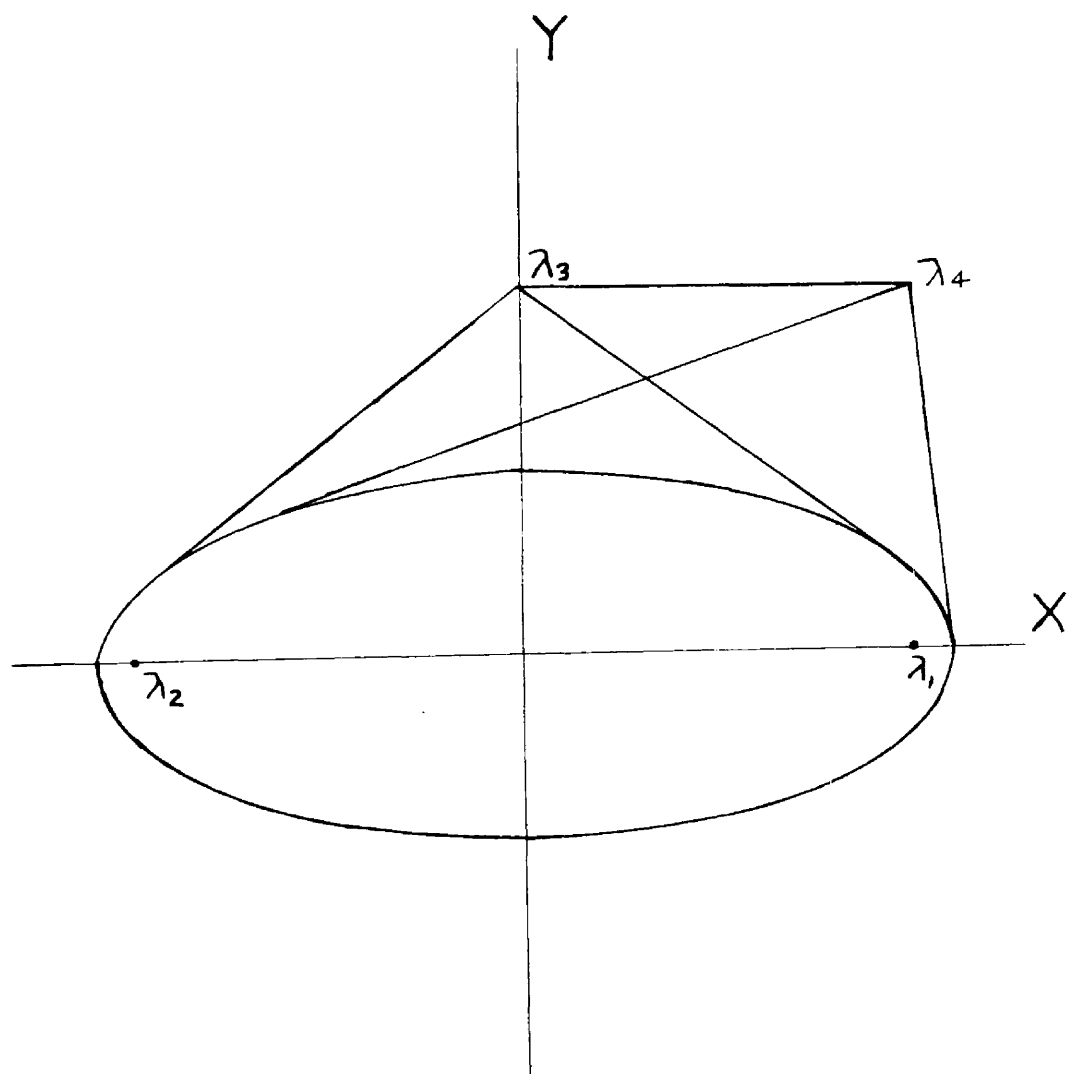


Figure 8.9

## IX

### Similar Matrices

1. The classical canonical form under the similarity group. Two matrices  $A$  and  $B$  are said to be similar if there exists a non-singular matrix  $P$  such that  $B = PAP^{-1}$ . The relationship is denoted by  $A \sim B$ , and the transformation  $PAP^{-1}$  is called the collineatory or similarity transformation. The study of the invariants under this transformation is of major importance in the theory of matrices. In particular, the following may be noted: A matrix  $A$  is similar to a matrix  $B$  if and only if  $\lambda I - A$  and  $\lambda I - B$  have the same invariant factors.<sup>1</sup> Among other things, similarity implies that the characteristic roots of  $A$  and  $B$  are identical.

A set of matrices is said to be a canonical set under the similarity group if: (1) every matrix is similar to a member of the set; (2) two distinct members of the set are not similar.<sup>2</sup> The existence of such canonical sets is well established; the one used here is the best one adapted to this study, since it is triangular. It is called in the literature the classical canonical set, and consists of matrices of the form

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<sup>1</sup> A. A. Albert, Modern Higher Algebra (Chicago: The University of Chicago Press, 1937), p. 77.

<sup>2</sup> Ibid., p. 76.

$$\begin{pmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ 0 & 0 & J_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & J_k \end{pmatrix}$$

where the  $J_i$  are Jordan matrices, to be described. Let the matrix be of order  $m$ , and have  $k$  distinct characteristic roots,  $\lambda_1, \dots, \lambda_k$ . Let the multiplicity of  $\lambda_i$  be  $m_i$ , so that  $\sum_{i=1}^k m_i = m$ . Then  $J_i$  is a matrix of order  $m_i$ :

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_i & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda_i & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \lambda_i & 0 \\ 0 & 0 & 0 & \dots & 0 & \lambda_i \end{pmatrix}$$

2. The effect of a similarity transformation on the field of values. The problem of this section is to determine what relationship, if any, there is between the fields of value of two similar matrices. The procedure is as follows: let  $C$  be a matrix of the canonical set, and let  $P$

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<sup>3</sup> H. W. Turnbull and A. C. Aitken, An Introduction to the Theory of Canonical Matrices (London: Blackie & Son Limited, 1932), pp. 61-62.

range over all non-singular matrices; then what sort of fields do the matrices  $PC_i P^{-1}$  have? Suppose  $P = (P_{ij})$ ,  $|P| = p$ ; then  $P^{-1} = (Q_{ij})$ , where  $Q_{ij} = p^{-1} P_{ij}$ ,  $P_{ij}$  being the cofactor of  $p_{ij}$ . A unitary matrix  $U$  is sought such that  $UPC_i P^{-1} \bar{U}'$  is triangular, in view of the greater ease of determining the nature of the field of values of a triangular matrix. In fact, in the study of the second and third order cases to follow, a matrix  $U$  is found such that  $UP$  is triangular, so that certainly the product  $UPC_i P^{-1} \bar{U}'$  is triangular.

3. Application to second order matrices. The classical canonical set for second order matrices has three members, which will be denoted by  $C_1, C_2, C_3$ . It may be assumed that they are in "standard position", for, if  $A \sim B$ , then  $A - kI \sim B - kI$  under the same transforming matrix  $P$ , and the effect on the field of values of each is simply a translation. Then the  $C_i$  are given by:

$$C_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad C_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad C_3 = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix}$$

More generally, write

$$C_i = \begin{pmatrix} c & q \\ 0 & -c \end{pmatrix}$$

where  $c = q = 0$  for  $C_1$ ,  $c = 0, q = 1$  for  $C_2$ , and  $c \neq 0, q = 0$



for  $C_3$ .

Let  $P = (p_{ij})$ ; then  $|P| = p_{11}p_{22} - p_{12}p_{21} \neq 0$ ,

and

$$P^{-1} = \begin{pmatrix} p_{22}/p & -p_{12}/p \\ -p_{21}/p & p_{11}/p \end{pmatrix}$$

Then the matrix  $U$  described above is:

$$U = \begin{pmatrix} \bar{p}_{11}/N & \bar{p}_{21}/N \\ -\bar{p}_{21}/N & p_{11}/N \end{pmatrix}$$

where  $N = \sqrt{p_{11}\bar{p}_{11} + p_{21}\bar{p}_{21}}$ . Calculation of  $UP$  and  $P^{-1}\bar{U}'$  yields:

$$UP = \begin{pmatrix} N & S/N \\ 0 & F/N \end{pmatrix} \quad P^{-1}\bar{U}' = \begin{pmatrix} 1/N & -S/pN \\ 0 & N/p \end{pmatrix}$$

where  $S = \bar{p}_{11}p_{12} + \bar{p}_{21}p_{22}$ . Now let  $D_i = UPC_iP^{-1}\bar{U}'$ ; then

$$D_i = \begin{pmatrix} c & (Nq - 2cS)/p \\ 0 & -c \end{pmatrix}$$

(1) For all  $P$ ,  $D_1 = C_1$ .

(2) For  $c = 0$ ,  $q = 1$ , there is obtained:

$$D_2 = \begin{pmatrix} 0 & N/p \\ 0 & 0 \end{pmatrix}$$

The field of values of  $D$  is a circle concentric with the circle which is the field of values of  $C_2$ , and with radius

$|N/2p|$ . Since for an appropriate choice of  $P$ , this radius can be given any positive value, it is concluded that the fields of value of the set of matrices similar to  $C_2$  consist of the family of circles concentric with the circle of  $C_2$ .

(3) For  $c \neq 0$ ,  $q = 0$ , there is obtained:

$$D_3 = \begin{pmatrix} c & -2cS/p \\ 0 & -c \end{pmatrix}$$

The field of values of  $D_3$  is an ellipse confocal with the ellipse which is the field of values of  $C_3$ , and with semi-minor axis  $|cS/p|$ . Since for an appropriate choice of  $P$ , this semi-minor axis can be given any positive value, it is concluded that the fields of value of the set of matrices similar to  $C_3$  consist of the family of ellipses confocal with the ellipse of  $C_3$ .

4. Application to third order matrices. The classical canonical set for third order matrices has six members, which will be denoted by  $C_1, C_2, \dots, C_6$ . As in the second order case, they are assumed to be in "standard position". Then the  $C_i$  are as follows:

$$C_1 = 0$$

$$C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C_4 = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -2c \end{pmatrix}$$

$$C_5 = \begin{pmatrix} c & 1 & 0 \\ 0 & c & 0 \\ 0 & 0 & -2c \end{pmatrix}$$

$$C_6 = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \quad \sum_{i=1}^3 c_i = 0, \quad c_i \neq c_j.$$

For the general part of the discussion, write

$$C_i = \begin{pmatrix} c_1 & q_1 & 0 \\ 0 & c_2 & q_1 \\ 0 & 0 & c_3 \end{pmatrix}$$

where  $\sum_{i=1}^3 c_i = 0$ , and  $q_i = 0$  or  $1$ .

Following the notation described above, the required matrix  $U$  for which  $UPC_1 P^{-1} \bar{U}'$  is triangular is:

$$U = \begin{pmatrix} \frac{\bar{p}_{11}}{N_1} & \frac{\bar{p}_{21}}{N_1} & \frac{\bar{p}_{31}}{N_1} \\ \frac{p_{21}\bar{p}_{33} - p_{31}\bar{p}_{32}}{N_2} & \frac{p_{31}\bar{p}_{31} - p_{11}\bar{p}_{33}}{N_2} & \frac{p_{11}\bar{p}_{32} - p_{21}\bar{p}_{31}}{N_2} \\ \frac{p_{31}}{N_3} & \frac{p_{32}}{N_3} & \frac{p_{33}}{N_3} \end{pmatrix}$$

where  $N_1 = \sqrt{p_{11}\bar{p}_{11} + p_{21}\bar{p}_{21} + p_{31}\bar{p}_{31}}$ ; and  $N_2$  and  $N_3$  in the same manner normalize the second and third rows. Incidentally,  $N_2 = N_1 N_3$ , Now calculate  $UP$  and  $P^{-1} \bar{U}'$ :

$$UP = \begin{pmatrix} N_1 & s_1/N_1 & s_2/N_1 \\ 0 & -N_3/N_1 & s_4/N_2 \\ 0 & 0 & p/N_3 \end{pmatrix}$$

$$P^{-1} \bar{U}' = \begin{pmatrix} 1/N_1 & s_1/N_2 & s_3/pN_3 \\ 0 & -N_1/N_3 & s_4/pN_3 \\ 0 & 0 & N_3/p \end{pmatrix}$$

where  $s_1 = \bar{p}_{11} p_{12} + \bar{p}_{21} p_{22} + \bar{p}_{31} p_{32}$

$$s_2 = \bar{p}_{11} p_{13} + \bar{p}_{21} p_{23} + \bar{p}_{31} p_{33}$$

$$s_3 = p_{11} \bar{p}_{31} + p_{22} \bar{p}_{32} + p_{33} \bar{p}_{33}$$

$$s_4 = p_{21} \bar{p}_{31} + p_{22} \bar{p}_{32} + p_{23} \bar{p}_{33}$$

Calculation of  $D_i = UPC_i P^{-1} \bar{U}$  then yields:

$$D_i = \begin{pmatrix} c_1 & d_{12} & d_{13} \\ 0 & c_2 & d_{23} \\ 0 & 0 & c_3 \end{pmatrix}$$

in which  $d_{12} = [(c_1 - c_2)S_1 - q_1 N_1^2] / N_3$ ,

$$d_{13} = \frac{(c_1 - c_2)N_1^2 S_3 + (c_3 - c_2)N_3^2 S_2 + q_1 N_1^2 S_4 + q_2 N_3 S_1}{pN_1 N_3}$$

$$d_{23} = [(c_3 - c_2)S_4 - q_2 N_3^2] / pN_1$$

(1)  $D_1 = C_1$  for all  $P$ .

(2)  $D_2$ . For  $c_1 = c_2 = c_3 = 0$ ,  $q_1 = 1$ ,  $q_2 = 0$ .

$$D_2 = \begin{pmatrix} 0 & -N_1^2 / N_3 & N_1 S_4 / pN_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$C(D_2)$  is a circle concentric with the circle obtained for  $C_2$ , plus the point  $\alpha = 0$ . The radius of the circle can be given any positive value for an appropriate choice of  $P$ , so that it is concluded that the family of curves  $C(D_2)$  consists of the family of circles with center at the origin, plus the point  $\alpha = 0$ .

(3)  $D_3$ . Here  $d_{12} = (-N_1^2) / N_3$ ,  $d_{13} = (N_1^2 S_4 + N_3^2 S_1) / pN_1 N_3$ ,  $d_{23} = (-N_3^2) / pN_1$ . Curves obtained consist of the circles described in (2), but with variable point  $\alpha$ ; the cardioid, and curves of Plücker type (a) (see table in section VII).

The existence of a triple characteristic root implies that the circular points at infinity are double points of  $C(D_3)$ .

(4)  $D_4$ . The matrix  $D_4$  is

$$D_4 = \begin{pmatrix} c & 0 & (-3cS_z N_3)/pN_1 \\ 0 & c & (-3cS_4)/pN_1 \\ 0 & 0 & -2c \end{pmatrix}$$

For these matrices,  $\alpha = c$ , the curves obtained are the family of confocal ellipses with foci at  $c$  and  $-2c$  (including the degenerate case of the straight line segment) plus the fixed  $\alpha = c$ .

(5)  $D_5$ . Here the double root implies that the curve  $C(D_5)$  passes through the circular points at infinity; the curves include (a) Circles with center at  $c$  plus tangents from  $\alpha$ , these tangents being real or imaginary according as  $\alpha$  is outside or inside the circle in a given case; (b) the confocal ellipses with foci at  $c$  and  $-2c$ , and tangents from  $\alpha$ , which may be real or imaginary, as in (a); the degenerate case of (a) is not obtained; (c) curves of Plucker type (a) and (b) of the table in section VII.

(6)  $D_6$ . The simplest way to describe this set is to point that  $C(D_6)$  does not pass through the circular points at infinity, so that the types which do are excluded; thus the circle and cardioid are excluded.

From the foregoing evidence obtained in the second and third order cases, it appears that to require that a matrix be similar to a particular member of a canonical set does effect some measure of restriction on the field of values, but for the most part the restriction is not very sharp.

## X

### Related Topics

1. A special type of matrix. Let  $A$  be a matrix  $(a_{ij})$  in which  $a_{ij} = 1$  when  $i < j$ , and  $a_{ij} = 0$  when  $i \geq j$ . Then, if  $tA = F_\theta + iG_\theta$ ,  $F_\theta$  has a characteristic root of multiplicity  $m - 1$  for  $\theta = 0^\circ$ , the root being  $-\frac{1}{2}$ , and the remaining root is  $(m - 1)/2$ . Thus the line  $x = -\frac{1}{2}$  is a multiple tangent of  $C(A)$ , tangent to it at  $m - 1$  distinct points. Between each two consecutive points of tangency there is a cusp with horizontal tangent, hence a total of  $m - 2$  cusps. Since a multiple tangent of order  $m - 1$  is equivalent to  $\frac{1}{2}(m - 1)(m - 2)$  bitangents, each represented in the equation of  $C(A)$  by the square of a linear factor, evidently the degree of  $C'(A)$  is  $m(m - 1) - (m - 1)(m - 2) = 2(m - 1)$ .

In particular, for  $m = 2$ , the curve  $C(A)$  is a circle with radius  $\frac{1}{2}$ ; for  $m = 3$ , the curve is a cardioid; for  $m = 4$ , see example 8.6.

2. On polygonal fields of value. In connection with normal matrices, whose fields of values are polygonal, the converse question arises: if  $A$  has a polygonal field of values, is  $A$  a normal matrix? The answer is evidently yes for  $m = 2, 3, 4$ , but for  $m = 5$ , an example shows that the answer is, in general, no. Let  $A$  be the matrix



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4\omega & 0 \\ 0 & 0 & 0 & 0 & 4\omega^2 \end{pmatrix}$$

Now  $C(A)$  consists of the triangle with vertices at  $4$ ,  $4\omega$ ,  $4\omega^2$ , the ellipse which is the field of values of the second order matrix in the upper left of  $A$ , and the tangents to this ellipse from the three points named. But since the ellipse is entirely within the triangle,  $W(A)$  consists of the triangle. Therefore, though  $A$  is not a normal matrix, it has a polygonal field of values.

### 3. On fields of value within a fixed rectangle.

For an arbitrary matrix  $A = F + iG$ , consider the set of matrices  $B = UF\bar{U}' + iVG\bar{V}'$ , where  $U$  and  $V$  range over the set of unitary matrices. These transformations leave the field of values and the characteristic roots of  $F$ , and of  $G$ , unchanged; therefore, for each  $B$ , the curve  $C(B)$  is tangent to each of  $m$  fixed vertical lines and  $m$  fixed horizontal lines.

In particular, let  $A$  be the second order matrix.

$$A = \begin{pmatrix} c & 2b \\ 0 & -c \end{pmatrix}$$

Then the field of values of  $A$  is the ellipse  $x^2/a^2 + y^2/b^2 = 1$ , and the fields of value of the matrices  $B$  are the ellipses tangent to all four sides of the rectangle  $x = a, x = -a, y = b, y = -b$ . The characteristic roots of the matrices  $B$ , which are the foci of the ellipses, lie on that portion of the rectangular hyperbola  $x^2 - y^2 = c^2$  which is inside or on the boundary of the rectangle. In the special case  $c = 0$ , the rectangle is a square, and these characteristic roots lie on the diagonals of the square.

4. Extension of the field of values concept. W. V. Parker has made some observations about the set  $x\bar{A}x'$ , where  $xPx' = 1$ ,  $P$  a positive definite Hermitian matrix. There is a unique positive definite Hermitian matrix  $Q$  such that  $Q\bar{Q}' = P$ .<sup>1</sup> Thus,  $xQ\bar{Q}'x' = 1$ , and if  $y = xQ$ , then  $y\bar{y}' = 1$ . For  $R = Q^{-1}$ , evidently  $x = yR$ , so that the set  $x\bar{A}x'$  is the set  $yR\bar{A}R'\bar{y}'$  where  $y\bar{y}' = 1$ . Hence the set  $x\bar{A}x'$  where  $xPx' = 1$  is the field of values of  $R\bar{A}R'$ .

More general point sets are obtained if the restriction of  $P$  to positive definite Hermitian be removed. As an illustration, let  $P$  be the second order matrix

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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<sup>1</sup> Macduffee, op. cit., p. 77.

Now by argument analogous to that used in section VI, the set  $xPA\bar{x}'$ , where  $xP\bar{x}' = 1$ , can be shown to be the set of points obtained by deleting from the plane the region between the two branches of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

No attempt has been made to generalize this type of set to higher orders.

The set  $xA\bar{y}'$  was completely described by W. V. Parker in 1937.<sup>2</sup> More recently, he has obtained results, not yet published, on this set under the restriction  $\bar{xy}' = 0$ . This latter set is evidently identical with the set of non-diagonal elements of all unitary transforms of A.

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<sup>2</sup> W. V. Parker, "The Characteristic Roots of Matrices," Duke Mathematical Journal, XII, (1945), 519-526.

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## AUTOBIOGRAPHY

John Cecil Currie was born in Oxford, Mississippi, on November 21, 1913. After brief residences in several places in Mississippi, South Carolina and Virginia, he entered school in Milford, Texas, in 1921. In 1924, he transferred to Hattiesburg, Mississippi, where he finished high school as valedictorian of the class of 1930. There followed three years at Mississippi State Teachers College, now Mississippi Southern College, also at Hattiesburg. After receiving his B. S. degree here in 1933, he spent a year teaching in a small rural high school in Mississippi. The next year he returned to school, entering the University of Mississippi, where he received his M. A. degree in June, 1936. In the fall of 1936, he entered Louisiana State University, spending one year as graduate fellow, a second as assistant, before accepting an instructorship at Northeast Junior College in Monroe, Louisiana. He spent eight years in this Monroe branch of the university, returning to Louisiana State University in 1946, on leave, to continue work on the degree of Doctor of Philosophy.

## EXAMINATION AND THESIS REPORT

Candidate: **John Cecil Currie**

Major Field: **Mathematics (Algebra)**

Title of Thesis: **The Field of Values of a Matrix**

Approved:

*W. V. Parker*

Major Professor and Chairman

*Wm O Serwaga*

Dean of the Graduate School

EXAMINING COMMITTEE:

*OB Summer*

*H. L. Smith*

*J. A. Riekey*

*George H. H. H.*

*H. Q. Runt*

Date of Examination: **May 7, 1948**